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The transformation law for the Szegő kernel under conformal mappings

To deduce the transformation laws for the Szegő projection and kernel under conformal mappings, we will require the following result.

Theorem 12.1. *Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded domains with C^∞ smooth boundaries. Then $f \in C^\infty(\overline{\Omega}_1)$ and f' is nonvanishing on $\overline{\Omega}_1$. Consequently, $f^{-1} \in C^\infty(\overline{\Omega}_2)$. Furthermore, f' is equal to the square of a function in $A^\infty(\Omega_1)$.*

The term *biholomorphic* means that f is a one-to-one holomorphic map of Ω_1 onto Ω_2 (and consequently f^{-1} is holomorphic on Ω_2). To prove this theorem, we will need the following lemma which will also prove to be useful later.

Lemma 12.1. *Suppose Ω is a bounded domain with C^∞ smooth boundary. Then Ω is biholomorphic to a bounded domain with real analytic boundary, i.e., there exists a bounded domain Ω_2 with real analytic boundary and a biholomorphic map of Ω onto Ω_2. The biholomorphic map extends C^∞ smoothly to the boundary and its derivative is nonvanishing on $\overline{\Omega}$.*

Proof of the lemma. This is a standard construction in conformal mapping. We proceed by induction on the connectivity of Ω. If Ω is 1-connected, i.e., simply connected, we use a Riemann mapping function to map Ω onto the unit disc (which has real analytic boundary). In this case, the lemma reduces to Theorem 8.2. Suppose the lemma has been established for $(n-1)$-connected domains, and suppose that Ω is n-connected. Let γ denote one of the *inner* boundary curves of Ω, i.e., one that bounds a bounded component of the complement of Ω in $\mathbb{C}$. Let Ω_{n-1} denote the $(n-1)$-connected domain obtained by filling in the hole in Ω bounded by γ, and let D^γ denote the domain enclosed by γ. By our induction hypothesis, there is a biholomorphic map G of Ω_{n-1} onto a domain $G(\Omega_{n-1})$ with real analytic boundary. Now, when G is restricted to Ω, we obtain a biholomorphic map of Ω onto the domain $G(\Omega)$ which is an n-connected domain such that all of its boundary curves but possibly one are real analytic. The one boundary curve that might not be real analytic is $G(\gamma)$, the image of γ under G. Let z_0 be a point in $G(D^\gamma)$. By composing G with the mapping $F(z) = 1/(z - z_0)$, we map Ω to a domain $F(G(\Omega))$ such that the inner boundaries are real analytic, and only the outer boundary might not be. Finally, to complete the induction, we use a Riemann map H that maps the simply connected domain obtained by filling in the holes of $F(G(\Omega))$ onto the unit disc. The mapping $H \circ F \circ G$ maps Ω into a subdomain of the unit disc. The inner boundaries are real analytic. The outer boundary is the unit circle, which is also real analytic. The statement about the smoothness up to the boundary of this map follows from the smoothness of the maps used in its construction (see Theorem 8.2). Similarly, the nonvanishing of the derivative of this map on $\overline{\Omega}$ follows from the nonvanishing of the derivatives of the maps used in its construction. The proof of the lemma is complete. □

Proof of Theorem 12.1. Using the lemma, let $h_1 : \Omega_1 \to G_1$ be a biholomorphic mapping of Ω_1 onto a domain G_1 with real analytic boundary and let $h_2 : \Omega_2 \to G_2$ be a biholomorphic mapping of Ω_2 onto a domain G_2 with real analytic boundary. Consider the biholomorphic map $H = h_2 \circ f \circ h_1^{-1}$ of G_1 onto G_2. Using Lemma 11.1, let $\Phi \in C^\infty(\overline{G_2})$ be a function that vanishes on bG_2 such that $\partial\Phi/\partial z = 1$ near bG_2, i.e., such that $1 - \partial\Phi/\partial z$ has compact support in G_2. We now claim that $\Phi \circ H$ extends continuously to $\overline{G_1}$ and that $\Phi \circ H$ is harmonic on G_1 near the boundary. Let $d_1(z)$ and $d_2(z)$ denote the distances from a point z to the boundaries of G_1 and G_2, respectively. It is easy to see that, given an $\epsilon > 0$, there is a $\delta > 0$ such that if a point $z \in G_1$ satisfies $d_1(z) < \delta$, then $d_2(H(z)) < \epsilon$. Indeed, if this were not the case, we could construct a sequence of points z_n in G_1 such that $d_1(z_n) \to 0$ as $n \to \infty$, but $d_2(H(z_n))$ remains bounded away from zero. By taking a subsequence, it would be possible to find a sequence of points z_n in G_1 converging to a boundary point $z_0 \in bG_1$ such that $H(z_n)$ converges to a point w_0 in G_2, implying that $H^{-1}(w_0) = z_0$, which is absurd. Since $\Phi = 0$ on bG_2, it follows that $\Phi \circ H$ is continuous up to bG_1 and is zero on the boundary. Since $\partial\Phi/\partial z = 1$ near bG_2, it follows that $(\partial\Phi/\partial z) \circ H = 1$ near bG_1. Because $\Delta = 4(\partial/\partial\bar{z})(\partial/\partial z)$, we can show that $\Phi \circ H$ is harmonic near bG_1 by showing that $(\partial/\partial z)(\Phi \circ H)$ is holomorphic near bG_1. Let us use subscript z's to indicate differentiation with respect to z. The chain rule yields that $(\Phi \circ H)_z = H'[\Phi_z \circ H]$. But, as mentioned above, $\Phi_z \circ H$ is equal to one near bG_1, and hence, $(\Phi \circ H)_z = H'$ near bG_1, which is holomorphic. Since $\Phi \circ H$ is continuous up to bG_1, zero on bG_1, and harmonic near bG_1, we may apply the classical Schwarz Reflection Principle to see that $\Phi \circ H$ extends past the boundary of G_1 as a harmonic function. Now, since $(\partial/\partial z)(\Phi \circ H) = H'$ near bG_1, we deduce that H' extends holomorphically past bG_1. Thus, it follows that $H \in C^\infty(\overline{G}_1)$ and we can say that $f = h_2 \circ H \circ h_1^{-1}$ is in $C^\infty(\overline{\Omega}_1)$. Since the same argument will show that $F = f^{-1}$ is in $C^\infty(\overline{\Omega}_2)$, it follows from the identity, $F'(f(z)) = 1/f'(z)$, that f' cannot vanish at any point in $\overline{\Omega}_1$.

To finish the proof, we must see that f' is the square of a function in $A^\infty(\Omega_1)$. Let $T_i(z)$ denote the unit tangent vector functions associated to $b\Omega_i$, $i = 1, 2$. Observe that, if $z(t)$ locally parameterizes the boundary of Ω_1 in the standard sense, then $\zeta(t) = f(z(t))$ locally parameterizes the boundary of Ω_2. As $z(t)$ traces out the boundary of Ω_1 in the standard sense, $f(z(t))$ traces out the boundary of Ω_2 *in the standard sense.* To see this, consider two unit vectors originating from a point $z \in b\Omega_1$, one pointing in the direction of $T_1(z)$, the other pointing in the direction of the inward normal vector. Since f is conformal in the interior, and since f is smooth up to the boundary, f satisfies the Cauchy-Riemann equations at z. The identity, $\zeta'(t) = f'(z(t))z'(t)$, yields that, as a linear map, $f'(z)$ maps $T_1(z)$ to a tangent vector at $f(z)$. Furthermore, since f maps Ω_1 into Ω_2, the inward pointing normal vector gets mapped to an inward pointing normal vector at $f(z)$. The fact that f preserves the sense of angles at z forces us to conclude that $\zeta'(t)$ points in the direction of the standard orientation. This shows that $f(z(t))$ traces out the boundary of Ω_2 in the standard sense. Hence, upon dividing $\zeta'(t) = f'(z(t))z'(t)$ by its modulus, we obtain the identity,

$$T_2(f(z)) = T_1(z)\frac{f'(z)}{|f'(z)|}. \tag{12.1}$$

Now, as any particular boundary component of Ω_i ($i = 1$ or 2) is traced out exactly once in the standard sense, the argument of T_i varies a total of $\pm 2\pi$. Hence, (12.1) reveals that the argument of $f'(z)$ varies a total of either zero or $\pm 4\pi$ on each boundary component of Ω_1. Since these numbers are all *even* multiples of 2π, we claim that it follows that $f'(z)$ has a single valued square root on Ω_1. Indeed, to see this, consider the variation of arg f' around any closed curve γ in Ω_1. The variation is given by the integral $\int_\gamma \frac{1}{i}\frac{f''}{f'}\,dz$. Because the boundary curves of Ω_1 form a homology basis for Ω_1, it follows that the integral is an even multiple of $\pm 2\pi$. This means that, if we were to analytically continue the germ of a square root for f' around γ, we would come back to our starting point with the original germ. Hence, f' has a single valued square root on Ω_1. Let $\sqrt{f'(z)}$ denote one of the square roots of f'. Note that equation (12.1) can now be rewritten

$$\overline{\sqrt{f'(z)}}T_2(f(z)) = \sqrt{f'(z)}T_1(z). \tag{12.2}$$

□

We may now state the transformation rule for the Szegő projection, the Szegő kernel, and the Garabedian kernel under biholomorphic maps. Let a subscript one or two indicate that the function or projection under discussion is associated to Ω_1 or Ω_2, respectively.

Theorem 12.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded domains with C^∞ smooth boundaries. The Szegő projections transform according to the formula*

$$P_1\left(\sqrt{f'}\,(\varphi \circ f)\right) = \sqrt{f'}\,((P_2\varphi) \circ f)$$

for all $\varphi \in L^2(b\Omega_2)$. The Szegő kernels transform according to

$$S_1(z,w) = \sqrt{f'(z)}\ S_2(f(z), f(w))\ \overline{\sqrt{f'(w)}}.$$

The Garabedian kernels transform according to

$$L_1(z,w) = \sqrt{f'(z)}\ L_2(f(z), f(w))\ \sqrt{f'(w)}.$$

The notation $\sqrt{f'}\,(\varphi \circ f)$ stands for: $\sqrt{f'}$ *times the quantity,* φ *composed with* f.

Proof. Define an operator Λ_1 that maps $C^\infty(b\Omega_2)$ into $C^\infty(b\Omega_1)$ via

$$\Lambda_1\varphi = \sqrt{f'}\,(\varphi \circ f).$$

It is clear that Λ_1 maps $A^\infty(\Omega_2)$ into $A^\infty(\Omega_1)$. It is also easy to verify that Λ_1 is isometric in L^2 norms, i.e., that

$$\|\Lambda_1\varphi\|_{L^2(b\Omega_1)} = \|\varphi\|_{L^2(b\Omega_2)}.$$

Indeed, because $f(z(t))$ parameterizes the boundary of Ω_2 in the standard sense when $z(t)$ parameterizes the boundary of Ω_1 in the standard sense, it follows that $ds_2 = |f'(z(t))z'(t)|\,dt = |f'|\,ds_1$ where ds_i denotes the differential element of arc length on $b\Omega_i$, $i = 1, 2$. Thus,

$$\int_{b\Omega_2} |\varphi|^2\,ds = \int_{b\Omega_1} |f'||\varphi \circ f|^2\,ds. \tag{12.3}$$

The inverse of Λ_1 can be written down in terms of the inverse of f. If we write $F = f^{-1}$, and if we choose the square root of F' that satisfies $\sqrt{F'(f(z))} = 1/\sqrt{f'(z)}$, then it is easy to check that the operator Λ_2 given by

$$\Lambda_2\psi = \sqrt{F'}\,(\psi \circ F)$$

is the inverse of Λ_1. Hence, it follows that Λ_1 extends uniquely to a Hilbert space isomorphism of $L^2(b\Omega_2)$ onto $L^2(b\Omega_1)$ which also restricts to be an isomorphism between the subspaces $H^2(b\Omega_2)$ and $H^2(b\Omega_1)$. We will now show that this fact alone implies the transformation rule, $P_1\Lambda_1 = \Lambda_1 P_2$, for the Szegő projections. Before we can begin, we recall the standard fact from Hilbert space theory that norm preserving operators must also preserve inner products. That this is so is easily seen by means of the polarization identity,

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) - \frac{i}{4}(\|u+iv\|^2 - \|u-iv\|^2).$$

Thus, we may assert that $\langle \Lambda_1 u, \Lambda_1 v\rangle_{b\Omega_1} = \langle u, v\rangle_{b\Omega_2}$ and similarly for Λ_2.

An important consequence of the fact that the operators Λ_1 and Λ_2 preserve the inner products is that

$$\langle \Lambda_1\varphi\,,\,\psi\rangle_{b\Omega_1} = \langle \varphi\,,\,\Lambda_2\psi\rangle_{b\Omega_2} \tag{12.4}$$

for all $\varphi \in L^2(b\Omega_2)$ and $\psi \in L^2(b\Omega_1)$. Indeed, since $\Lambda_1\Lambda_2$ is the identity, we may write

$$\langle \Lambda_1\varphi\,,\,\psi\rangle_{b\Omega_1} = \langle \Lambda_1\varphi\,,\,\Lambda_1\Lambda_2\psi\rangle_{b\Omega_1} = \langle \varphi\,,\,\Lambda_2\psi\rangle_{b\Omega_2}.$$

We now claim that if $u \in H^2(b\Omega_2)^\perp$, then $\Lambda_1 u \in H^2(b\Omega_1)^\perp$. To see this, suppose that $h \in H^2(b\Omega_1)$ and $u \in H^2(b\Omega_2)^\perp$. Then (12.4) yields

$$\langle \Lambda_1 u, h\rangle_{b\Omega_1} = \langle u, \Lambda_2 h\rangle_{b\Omega_2} = 0$$

because $\Lambda_2 h \in H^2(b\Omega_2)$. This proves our claim. Now the transformation rule follows by decomposing a function $\varphi \in L^2(b\Omega_2)$ as $\varphi = h + u$ where $h = P_2\varphi$ and $u = P_2^\perp\varphi$. Then $\Lambda_1\varphi = \Lambda_1 h + \Lambda_1 u$ is an orthogonal sum and the transformation formula rule follows because $P_1\Lambda_1\varphi = \Lambda_1 h = \Lambda_1 P_2\varphi$.

We now turn to the proof of the transformation formulas for the kernels. Let $S_a(z) = S_2(z, a)$. If $h \in H^2(b\Omega_1)$, then using (12.4), we may write

$$\langle h, \Lambda_1 S_a\rangle_{b\Omega_1} = \langle \Lambda_2 h, S_a\rangle_{b\Omega_2} = (\Lambda_2 h)(a).$$

Furthermore, if $G(z) = \overline{\sqrt{F'(a)}}\,S_1(z, F(a))$, then

$$\langle h, G\rangle_{b\Omega_1} = (\Lambda_2 h)(a).$$

Thus $\langle h, \Lambda_1 S_a\rangle_{b\Omega_1} = \langle h, G\rangle_{b\Omega_1}$ for all $h \in H^2(b\Omega_1)$, and we must conclude that $\Lambda_1 S_a = G$, i.e., that

$$\sqrt{f'(z)}S_2(f(z), a) = S_1(z, F(a))\overline{\sqrt{F'(a)}},$$

which is equivalent to the transformation rule for the Szegő kernel stated in the theorem. The formula for the Garabedian kernel follows from the Szegő kernel formula and identities (7.1) and (12.2). The proof is finished. □

The transformation formula for the Szegő kernels under a biholomorphic map gives rise to another nice formula for the Riemann mapping function.

Theorem 12.3. *Suppose that Ω is a bounded simply connected domain with C^∞ smooth boundary and suppose that f is a biholomorphic mapping from Ω onto the unit disc U such that for a certain point a in Ω, we have $f(a) = 0$ and $f'(a) > 0$. Then*

$$f'(z) = 2\pi \frac{S(z,a)^2}{S(a,a)}.$$

Proof. The proof of this identity rests on the simple fact that the Szegő kernel $S_U(z,w)$ for the unit disc satisfies $S_U(z,0) \equiv 1/(2\pi)$, which can be read off from Theorem 7.2. Hence, the transformation formula yields

$$\frac{1}{2\pi}\sqrt{f'(z)} = \sqrt{f'(z)}S_U(f(z),0) = cS(z,a)$$

where $\bar{c} = [\sqrt{f'(a)}]^{-1}$. Since $f'(a) > 0$, we may assume that we are dealing with a square root function that makes c a positive real number. Plugging $z = a$ into this formula yields that $f'(a) = 2\pi S(a,a)$. Now, we may square the formula and replace $f'(a)$ by its expression in terms of the Szegő kernel to obtain the formula for the derivative of the Riemann map. □

We remark that another, and more classical, way to think about the Szegő kernel is in terms of an orthonormal basis for $H^2(b\Omega)$. Suppose that $\{h_i\}_{i=1}^\infty$ is such a basis. We will prove that

$$S(z,a) = \sum_{i=1}^{\infty} h_i(z)\overline{h_i(a)}, \tag{12.5}$$

with absolute and uniform convergence in z on compact subsets of Ω. Granted this fact, the transformation formula for the Szegő kernel is seen to be a direct consequence of the fact that the operator Λ_1 sends an orthonormal basis for $H^2(b\Omega_2)$ to an orthonormal basis for $H^2(b\Omega_1)$.

To prove (12.5), notice that the coefficients c_i in the orthogonal expansion $S_a = \sum c_i h_i$ for the Szegő kernel are given by $c_i = \langle S_a, h_i\rangle_b = \overline{h_i(a)}$. The Cauchy integral formula $h(w) = \langle h, C_w\rangle_b$ gives rise to the basic estimate $|h(w)| \leq \|h\|\|C_w\|$ which shows that convergence in $H^2(b\Omega)$ implies uniform convergence on compact subsets of Ω. Hence, $\sum_{i=1}^\infty h_i(z)\overline{h_i(a)}$ converges in z uniformly on compact subsets of Ω to $S_a(z)$. The absolute convergence of the series follows from the observation that

$$S(a,a) = \sum_{i=1}^{\infty} |h_i(a)|^2.$$

Hence, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\sum_{i=1}^{\infty} |h_i(z)h_i(a)| &\leq \left(\sum_{i=1}^{\infty} |h_i(z)|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |h_i(a)|^2\right)^{1/2} \\ &= \sqrt{S(z,z)}\sqrt{S(a,a)}\end{aligned}$$

and absolute convergence is proved.

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The Ahlfors map of a multiply connected domain

Suppose Ω is a bounded simply connected domain. Everyone knows that, among all holomorphic functions h that map Ω into the unit disc, the Riemann mapping function associated to a point $a \in \Omega$ is the unique function in this class making $h'(a)$ real and as large as possible. Hence, finding the Riemann map is equivalent to solving an extremal problem. In Chapter 8, we showed that the solution to this extremal problem can also be expressed as the quotient of the Szegő and the Garabedian kernels, $f(z) = S(z,a)/L(z,a)$. In this chapter, we will consider this quotient when Ω is a *multiply connected* domain. We will show that it is a mapping of the domain onto the unit disc, that it solves the same extremal problem, and that it has many of the geometric features one would expect of a "Riemann mapping function" of a multiply connected domain. The map is known as the Ahlfors mapping.

Because we will be studying the extremal problem mentioned above, let us spell it out. Let $\mathcal{F}$ denote the set of holomorphic functions on Ω mapping Ω into the unit disc.

The Extremal Problem. *Given $a \in \Omega$, find all the functions $h \in \mathcal{F}$ that maximize $|h'(a)|$.*

To study the Ahlfors map, we will need to know a generalized version of the argument principle that allows zeroes to occur on the boundary. In the discussion of the argument principle that follows, we will allow functions to have zeroes *and poles* on the boundary because we will need this version later in the book. Let Ω be a bounded domain with C^∞ smooth boundary. Suppose that h is meromorphic in a neighborhood of $\overline{\Omega}$, and that h is not identically zero. Then the zeroes and poles of h are isolated. Let $\{z_i\}_{i=1}^N$ denote the zeroes of h that lie in Ω, let $\{p_i\}_{i=1}^Q$ denote the poles of h that lie in Ω, let $\{b_i\}_{i=1}^M$ denote the zeroes of h that lie on $b\Omega$, and let $\{B_i\}_{i=1}^R$ denote the poles of h that lie on $b\Omega$. For small $\epsilon > 0$, let $\gamma_\epsilon = b\Omega - \left(\cup_{i=1}^M D_\epsilon(b_i)\right) - \left(\cup_{i=1}^R D_\epsilon(B_i)\right)$. We assume that ϵ is small enough so that the closures of all the discs $D_\epsilon(b_i)$ and $D_\epsilon(B_j)$ are mutually disjoint and the set γ_ϵ consists of finitely many connected smooth arcs. On each of these arcs, the increment of the argument of

$h(z)$ as z moves along the arc in the positive sense is well defined. We will prove that as $\epsilon \to 0$, the sum of the increments of the argument of $h(z)$ along all these arcs tends to an angle $\Delta\arg h$, and this angle is related to the number of zeroes and poles of h according to the following formula (we let $m_h(z)$ denote the multiplicity of a zero or pole of h at z).

The generalized argument principle

$$\sum_{i=1}^{N} m_h(z_i) + \frac{1}{2}\sum_{i=1}^{M} m_h(b_i) - \sum_{i=1}^{Q} m_h(p_i) - \frac{1}{2}\sum_{i=1}^{R} m_h(B_i) = \frac{1}{2\pi}\Delta\arg h$$

In words, this generalized argument principle says that zeroes or poles in the interior of Ω contribute to the increment of the argument of h as usual, whereas zeroes and poles on the boundary contribute half the normal amount to the increment. To prove this fact, let $C_\epsilon(b_i)$ denote the arc of the circle that bounds $D_\epsilon(b_i)$ lying outside of Ω and let $C_\epsilon(B_i)$ denote the arc of the circle that bounds $D_\epsilon(B_i)$ lying outside of Ω. We assume that these circular arcs are parameterized so that $\gamma_\epsilon \cup \left(\cup_{i=1}^{M} C_\epsilon(b_i)\right) \cup \left(\cup_{i=1}^{R} C_\epsilon(B_i)\right)$ represents the boundary of

$$\Omega_\epsilon = \Omega \cup \left(\cup_{i=1}^{M} D_\epsilon(b_i)\right) \cup \left(\cup_{i=1}^{R} D_\epsilon(B_i)\right),$$

parameterized in the standard sense. We may now apply the classical argument principle to h on Ω_ϵ. As we let ϵ tend to zero, the increment of arg h on $C_\epsilon(b_i)$ is easily seen to approach $\pi m_h(b_i)$, and the increment of arg h on $C_\epsilon(B_i)$ is seen to approach $-\pi m_h(B_i)$. This completes the proof. We may now prove the following theorem.

Theorem 13.1. *Suppose Ω is a bounded n-connected domain with C^∞ smooth boundary and let $a \in \Omega$ be given. Then $L_a(z)$ is nonvanishing for $z \in \overline{\Omega} - \{a\}$. The function $S_a(z)$ is nonvanishing on $b\Omega$ and has exactly $n-1$ zeroes in Ω. The function $f(z) = S_a(z)/L_a(z)$ maps Ω onto the unit disc and is an n-to-one map (counting multiplicities). Among all holomorphic functions h that map Ω into the unit disc, the functions that maximize the quantity $|h'(a)|$ are given by $e^{i\theta} f(z)$ for some real constant θ. Furthermore, f is uniquely characterized as the solution to this extremal problem such that $f'(a) > 0$. Also, f extends to be in $C^\infty(\overline{\Omega})$, f' is nonvanishing on the boundary, and f maps each boundary curve one-to-one onto the boundary of the unit disc.*

Proof. Proving this theorem is easiest when Ω is a bounded finitely connected domain with *real analytic* boundary; so that is what we now assume. Later, we will relax this hypothesis. With the real analytic assumption, we know by Theorem 11.2 that $S_a(z)$ and $L_a(z)$ extend holomorphically past the boundary. This fact, together with identity (7.1), is ninety percent of the proof. The other ten percent is contained in the generalized argument principle proved above.

The function f has a simple zero at $z = a$ because $S(a,a) > 0$ and L_a has a simple pole at $z = a$. Besides this fact, at the moment, we know only that f is a meromorphic function on a neighborhood of $\overline{\Omega}$. Let λ be a complex number of unit modulus. We want to consider how many times $f(z)$ assumes the value λ on $\overline{\Omega}$. To do this, let $G(z) = S_a(z) - \lambda L_a(z)$. We will first prove that $G(z)$ has exactly n zeroes on $\overline{\Omega}$, one on each boundary component of Ω. Observe that formula (7.1) implies that, on $b\Omega$,

$$GT = S_aT - \lambda L_aT = i\ \overline{L_a} - \lambda i\ \overline{S_a} = -i\lambda(\overline{S_a} - \overline{\lambda}\overline{L_a}) = -i\lambda\overline{G}.$$

Thus, we have the identity,

$$G^2 = -i\lambda|G^2|\overline{T}. \tag{13.1}$$

Let $\{\gamma_i\}_{i=1}^n$ denote the simple closed real analytic curves that represent the n boundary components of Ω. We will now show that G has at least one zero on each γ_i. Indeed, if G has no zero on γ_i, identity (13.1) shows that the increment of arg G^2 around γ_i is the same as the increment of arg $\overline{T}$. But the increment of the argument of G^2, the *square* of a holomorphic function, around γ_i is either zero or an *even* multiple of $\pm 2\pi$, and the increment of arg $\overline{T}$ is $\pm 2\pi$. Hence, equality is out of the question. Thus, G must have at least one zero on γ_i.

Let $\{z_i\}_{i=1}^N$ denote the zeroes of G that lie in Ω and let $\{b_i\}_{i=1}^M$ denote the zeroes of G that lie on $b\Omega$. Observe that G has a single simple pole at a. Thus, the generalized argument principle yields

$$-1 + \sum_{i=1}^{N} m_G(z_i) + \frac{1}{2}\sum_{i=1}^{M} m_G(b_i) = \frac{1}{2\pi}\Delta\text{arg } G.$$

Now, identity (13.1) reveals that

$$\frac{1}{2\pi}\Delta\text{arg } G^2 = \frac{1}{2\pi}\Delta\text{arg } \overline{T} = -1 + (n-1) = n-2.$$

Therefore, $\frac{1}{2\pi}\Delta\text{arg } G = \frac{1}{2}(n-2)$. But the term $\sum_{i=1}^M m_G(b_i)$ is at least n because G has at least one zero on each boundary component of Ω. When these numbers are plugged into the argument principle, we are forced to conclude that G has no zeroes in Ω, and *exactly* one zero on each component $b\Omega$.

Using what we know about G, we will now show that f extends to be holomorphic on a neighborhood of $\overline{\Omega}$, that $|f| < 1$ on Ω, and that $|f| = 1$ on $b\Omega$. Indeed, if f had a pole at some point in $\overline{\Omega}$, then there would exist a point $p \in \Omega$ near the pole such that $|f(p)| > 1$ and $L_a(p) \neq 0$. Consider the function $|f(z)|$ along a curve in Ω joining p to a that does not pass through a zero of L_a. Since $|f(p)| > 1$ and $|f(a)| = 0$, the intermediate value theorem implies that there is a point $z_0 \in \Omega$ (which is not a zero of L_a) such that $f(z_0) = \lambda$, a complex number of unit modulus. However, at such a point z_0, the function G associated to this λ that we studied above would have to vanish, and we have shown that it cannot vanish at an interior point. Hence, f has no poles in $\overline{\Omega}$. Now, if there is a point p in $\overline{\Omega}$ with $|f(p)| > 1$, we may apply the same reasoning to obtain the same contradiction. Hence, $|f| \leq 1$ on Ω. Since $f(a) = 0$, the maximum principle implies that $|f| < 1$ on Ω. Identity (7.1) yields that $|f| = 1$ on the dense subset of $b\Omega$ where L_a is nonvanishing. By continuity, $|f| = 1$ on all of $b\Omega$.

Next, we show that L_a is nonvanishing on $\overline{\Omega} - \{a\}$. We have shown that if $|\lambda| = 1$, then $G_\lambda = S_a - \lambda L_a$ has exactly one zero on each boundary component of Ω and no zeroes inside Ω. Also, because $f = S_a/L_a$ is holomorphic in a neighborhood of $\overline{\Omega}$, it follows that if $L_a(z_0) = 0$ with $z_0 \in \overline{\Omega}$, then $S_a(z_0) = 0$ too. Hence, G_λ and S_a must vanish wherever L_a does. Because G_λ cannot vanish in Ω, this yields that L_a cannot vanish in Ω. To see that L_a cannot vanish on $b\Omega$ either, suppose $L_a(z_0) = 0$, $z_0 \in b\Omega$. Then $S_a(z_0) = 0$, and $G_\lambda(z_0) = 0$ for *any* λ of unit modulus. We have shown that S_a cannot be identically zero on Ω; thus there is a point ξ_0 in the *same* boundary component of $b\Omega$ that z_0 is in such that $S_a(\xi_0) \neq 0$. Since $S_a(z_0) = 0$, it follows that $\xi_0 \neq z_0$. Formula (7.1) shows that $|S_a(\xi_0)| = |L_a(\xi_0)|$. Hence, there is a λ with $|\lambda| = 1$ such that $G_\lambda(\xi_0) = 0$. We have now shown that, for this particular choice of λ, G_λ has zeroes at two points, z_0 and ξ_0, in a single boundary curve. This is a contradiction. Therefore, L_a is nonvanishing on $\overline{\Omega} - \{a\}$. Formula (7.1) now shows that S_a is nonvanishing on $b\Omega$ too. We can also read off from (7.1) that

$$-\Delta\text{arg } S_a = \Delta\text{arg } L_a + \Delta\text{arg } T = 2\pi[-1 + 1 - (n-1)].$$

Hence, $\Delta\text{arg } S_a = 2\pi(n-1)$, and we deduce that S_a has exactly $n-1$ zeroes in Ω, and, because of the pole of L_a at a, that f has exactly n zeroes in Ω.

Since L_a is nonvanishing, and since G_λ vanishes exactly once on each boundary component of Ω when $|\lambda| = 1$, it follows that f maps each boundary component one-to-one onto the unit circle. This fact alone could be used to see that f is an n-to-one map (counting multiplicities) of Ω onto the unit disc. We will use the following simple argument to prove this instead. Given w in the unit disc, the number of times that f assumes the value w (counting multiplicities) on Ω is given by the integral

$$M(w) = \frac{1}{2\pi i}\int_{b\Omega} \frac{f'(z)}{f(z)-w}\, dz.$$

We know that $M(0) = n$. Since $M(w)$ is a holomorphic function of w on the unit disc taking values in the integers, M must be constant, i.e., $M(w) \equiv n$.

Finally, we must see that f is extremal, i.e., that, among all holomorphic functions on Ω that map into the unit disc, f has the property that $|f'(a)|$ is as large as possible. First, note that $f'(a) = 2\pi S(a,a)$ because $L(z,a)$ has a simple pole at $z = a$ with residue $1/2\pi$. Since multiplication by $e^{i\theta}$ preserves the class of functions that map into the unit disc, and since $f'(a) > 0$, we may restrict our attention to the class of functions h that map into the disc such that $h'(a) > 0$. Since this class is a normal family, we know extremal functions exist. Furthermore, if h is extremal, it must be that $h(a) = 0$. Indeed, if this is not the case, by forming the composition $M \circ h$ where $M(z)$ is the Möbius transformation $M(z) = (z - h(a))/(1 - \overline{h(a)}\, z)$, we obtain a map in the class with strictly larger derivative at a, which contradicts the extremal assumption. Hence, we may restrict our attention to the class $\mathcal{F}_+$ of functions h that are holomorphic on Ω, mapping Ω into the unit disc, such that $h(a) = 0$ and $h'(a) > 0$. Note that, by Theorem 6.4, $\mathcal{F}_+$ may be viewed as a subset of $H^2(b\Omega)$.

Consider the function L_a^2. It is meromorphic on a neighborhood of $\overline{\Omega}$ and has a single pole at $z = a$. We now claim that the residue of L_a^2 at a is zero. Indeed, by (7.1), $L_a^2 T = iL_a\overline{S_a}$. Hence, the residue of L_a^2 at a can be computed via

$$2\pi i\,\mathrm{Res}_a L_a^2 = \int_{b\Omega} L_a^2 T\, ds = \int_{b\Omega} iL_a\overline{S_a}\, ds = \langle iL_a, S_a\rangle_b,$$

and this last quantity is zero because $L_a = i\,\overline{S_a T}$ is orthogonal to $H^2(b\Omega)$. Hence,

$$L_a^2 = \frac{1}{4\pi^2}\frac{1}{(z-a)^2} + H_a$$

where H_a is holomorphic on a neighborhood of $\overline{\Omega}$. Hence, by the residue theorem, if $h \in \mathcal{F}_+$, then

$$h'(a) = 4\pi^2 \mathrm{Res}_a(L_a^2 h) = \frac{2\pi}{i}\int_{b\Omega} L_a^2 hT\ ds \le 2\pi \int_{b\Omega} |L_a^2|\ ds \tag{13.2}$$

since $|h| \le 1$ on $b\Omega$. But, by (7.1), the $L^2(b\Omega)$ norm of L_a is equal to that of S_a, which we know is equal to $S(a,a)^{1/2}$. Hence, we have shown that $h'(a) \le 2\pi S(a,a) = f'(a)$, and therefore, that f is an extremal function.

We next show that f is the *unique* extremal function in the class $\mathcal{F}_+$. This turns out to be an easy consequence of a measure theory exercise which asserts that, if $|v| \le |u|$ and $\int v = \int |u|$, then $v = |u|$. Suppose that $h \in \mathcal{F}_+$ is extremal, and let $v = -iL_a^2 hT$ and $u = |L_a|^2$. Because h maps into the unit disc, it follows from Theorem 6.4 that $|v| \le |u|$ on $b\Omega$. Now the reasoning used to deduce (13.2) shows that $\int v = \int |u|$ and it follows that $v = |u|$, i.e., that $-iL_a^2 hT = |L_a|^2$. Solving this equation for h shows that $h = i\,\overline{L_a T}/L_a$ and using (7.1) yields that $h = S_a/L_a$ as desired.

To finish the proof in the case of real analytic boundary, we must show that f' is nonvanishing on the boundary. Suppose $z_0 \in b\Omega$ and let u denote the harmonic function on Ω given by $u(z) = \text{Re}\,\overline{f(z_0)}\,f(z)$. This function is in $C^\infty(\overline{\Omega})$ and assumes its maximum value of one on $\overline{\Omega}$ at the boundary point z_0. The Hopf lemma (Theorem 9.4) states that the normal derivative of u at z_0 is nonzero. The chain rule now implies that $f'(z_0)$ cannot vanish either. The proof is complete for domains with real analytic boundaries.

To prove the theorem in the general case, we use Lemma 12.1. Let $G : \Omega \to \Omega_2$ be a biholomorphic map of our C^∞ smooth domain Ω onto a domain Ω_2 with real analytic boundary. We know that $G \in C^\infty(\overline{\Omega})$ and that G' is nonvanishing on $\overline{\Omega}$. Given $a \in \Omega$, it is clear that the solution to the extremal problem for Ω at a is given by $f = e^{i\theta} f_2 \circ G$ where f_2 is the solution to the extremal problem for Ω_2 at $G(a)$ with $f_2'(G(a)) > 0$ and $e^{i\theta}$ is a complex number of unit modulus chosen so that $f'(a) > 0$. To be precise,

$$e^{i\theta} = \overline{G'(a)}/|G'(a)| = \overline{\sqrt{G'(a)}}/\sqrt{G'(a)}.$$

We know that $f_2(z) = S_2(z, G(a))/L_2(z, G(a))$. Now, using the transformation formulas for the Szegő and Garabedian kernels under G, we obtain $f = S(z,a)/L(z,a)$ and the proof is finished. $\square$

14

The Dirichlet problem in multiply connected domains

We have postponed the study of the Dirichlet problem in multiply connected domains until now because, before Theorem 13.1, we did not know that the Garabedian kernel was nonvanishing on a multiply connected domain, and this is an important ingredient in our approach to the problem.

Life in a multiply connected domain is complicated by the fact that not every harmonic function can be globally written as the sum of a holomorphic and an antiholomorphic function. In order to obtain a theorem analogous to Theorem 10.1 in a multiply connected domain, we will have to add some terms to put the functions involved in the space of functions $\{h + \overline{H} : h, H \in H^2(b\Omega)\}$.

Theorem 4.3 implies that the operator that sends a function $\varphi \in C^\infty(b\Omega)$ to the function in $C^\infty(b\Omega)$ given by

$$h + \overline{T}\,\overline{H}$$

where $h = P\varphi$ and $H = P(\overline{T}\overline{\varphi})$ is the identity operator. Thus, we may decompose $S_a\,\varphi$ on $b\Omega$ as

$$S_a\,\varphi = h + \overline{T}\,\overline{H}$$

where $h = P(S_a\varphi)$ and $H = P(\overline{TS_a\varphi})$. Next, using (7.1), we substitute $-iS_a/\overline{L_a}$ for $\overline{T}$ and divide the identity by S_a. What we get is a decomposition of φ on the boundary of Ω as $h + \overline{H}$ where

$$h = \frac{P(S_a\,\varphi)}{S_a} \qquad \text{and} \qquad H = i\frac{P(\overline{S_a\varphi T})}{L_a}.$$

Using (7.1) again, we may replace $\overline{S_aT}$ in the expression for H by $-iL_a$. We have proved most of the statements in the following theorem.

Theorem 14.1. *Suppose Ω is a bounded finitely connected domain with C^∞ smooth boundary and suppose φ is a function in $C^\infty(b\Omega)$. Let $a \in \Omega$ be given. Then, on the boundary, the function φ can be decomposed as*

$$\varphi = h + \overline{H},$$

where h is a meromorphic function on Ω that extends C^∞ smoothly up to $b\Omega$ given by

$$h = \frac{P(S_a\,\varphi)}{S_a}$$

and H is a holomorphic function in $A^\infty(\Omega)$ given by

$$H = \frac{P(L_a\,\overline{\varphi})}{L_a}.$$

Furthermore, if φ is equal to $g + \overline{G}$ for some g and G in $H^2(b\Omega)$, then the function h has no poles and is in $A^\infty(\Omega)$. Thus, in this case, $h + \overline{H}$ is the harmonic extension of φ to Ω.

Proof. The decomposition of φ on the boundary as $h + \overline{H}$ was proved above. To see that h is meromorphic, we use the fact that Szegő kernel $S_a(z)$ has exactly $n-1$ zeroes in Ω when Ω is an n-connected domain, and $S_a(z)$ does not vanish for any $z \in b\Omega$ (see Theorem 13.1). The smoothness up to the boundary follows from the fact that P preserves $C^\infty(b\Omega)$ and the fact that $S_a(z)$ is in $A^\infty(\Omega)$. The function H is in $A^\infty(\Omega)$ because $L_a(z)$ has a single simple pole at a and does not vanish for any $z \in \overline{\Omega} - \{a\}$, and furthermore, L_a extends C^∞ smoothly up to the boundary.

Next, we show that if $\varphi \in C^\infty(b\Omega)$ can be decomposed as $\varphi = g + \overline{G}$ where g and G are in $H^2(b\Omega)$, then g and G must be in $A^\infty(\Omega)$. Note that if $\varphi = g + \overline{G}$, then, using (7.1), we see that $S_a\varphi = S_a g + i\,\overline{GL_aT}$. By subtracting $G(a)$ from G and adding $\overline{G(a)}$ to g, we may assume that $G(a) = 0$. In this case, it follows that GL_a is in $H^2(b\Omega)$. Hence, $S_a g + i\,\overline{GL_aT}$ is an orthogonal decomposition of $S_a\varphi$. It follows that $S_a g = P(S_a\varphi)$ and this shows that $g \in C^\infty(b\Omega)$. Consequently, $g \in A^\infty(\Omega)$. In fact, we have shown that $g = P(S_a\varphi)/S_a$ on the boundary, i.e., that $g = h$ on $b\Omega$. Since a holomorphic function that is continuous up to the boundary cannot vanish on an open set in the boundary, it follows that g must be equal to h on Ω, too. Hence, h has no poles in Ω. Since $g = h$, it follows that $G = H$, and the proof is complete. □

If Ω is n-connected, let $\{b_i\}_{i=1}^{n-1}$ be a set of points comprised of one point from each of the bounded connected components of the complement of $\overline{\Omega}$ in $\mathbb{C}$, and let γ_i denote the boundary curve of Ω bounding the component of $\mathbb{C} - \Omega$ containing b_i. The function

$$\psi_i = \log|z - b_i|$$

is a harmonic function on a neighborhood of $\overline{\Omega}$. We claim that ψ_i cannot be expressed as the real part of a single valued holomorphic function on

Ω. Indeed, locally, ψ_i has a harmonic conjugate given by a continuous choice of an argument of $z - b_i$ plus a constant. If such a local conjugate is continued around the curve γ_i in the standard sense, a simple computation based on the argument principle shows that the ending value differs from the starting value by -2π. Hence, ψ_i cannot have a single valued harmonic conjugate on Ω. Also note that the variation of a local harmonic conjugate for ψ_i around γ_j where $j \neq i$ is zero. We may repeat the argument above to deduce that if ψ, given by $\psi = \sum_{i=1}^{n-1} c_i\psi_i$, is the real part of a holomorphic function on Ω, then all the c's must be zero.

We have shown that ψ_i cannot be written on $b\Omega$ as $g_i + \overline{G_i}$ for any $g_i, G_i \in H^2(b\Omega)$. It follows that if we express ψ_i on $b\Omega$ as $h_i + \overline{H_i}$ where h_i and H_i are given by the formulas in Theorem 14.1, then h_i must have a pole at one of the zeroes of the Szegő kernel. In fact, the same reasoning shows that if $\{c_i\}_{i=1}^{n-1}$ are constants that are not all zero, then $\psi = \sum_{i=1}^{n-1} c_i\psi_i$ cannot be written as the real part of a holomorphic function on Ω, and therefore cannot be written on $b\Omega$ as $g + \overline{G}$ where $g, G \in H^2(b\Omega)$. This means that $\sum c_ih_i$ has at least one pole at a zero of the Szegő kernel.

Let $\{a_j\}_{j=1}^{n-1}$ denote the $n-1$ zeroes of $S_a(z)$ in Ω. For simplicity, let us suppose, for the moment, that each zero has multiplicity one. (In fact, we will prove in Chapter 27 that the zeroes of S_a become simple zeroes as a tends to the boundary of Ω. Hence, it is actually possible to choose a so that this condition is met.) The fact that $\sum c_ih_i$ has at least one pole at a zero of the Szegő kernel if not all the c_i's are zero means that if

$$\sum_{i=1}^{n-1} c_i \left(P(\psi_i S_a)\right)(a_j) = 0 \qquad \text{for } j = 1, \ldots, n-1,$$

then $c_i = 0$ for all i. This implies that the determinant of this linear system is nonzero. Hence, given $\varphi \in C^\infty(b\Omega)$, we may solve the linear system,

$$\sum_{i=1}^{n-1} c_i \left(P(\psi_i S_a)\right)(a_j) = P(\varphi S_a)(a_j),$$

$j = 1, \ldots, n-1$, for c_i, $i = 1, \ldots, n-1$. Having solved the system, we deduce that, when $\varphi - \sum c_i\psi_i$ is expressed as $h + \overline{H}$ via the formulas in Theorem 14.1, the function h has no poles at the zeroes of S_a. Hence, the harmonic extension of φ to Ω is given by

$$h + \overline{H} + \sum c_i \log|z - b_i|.$$

Now, in the case that $S_a(z)$ has zeroes of multiplicity greater than one, we must solve a linear system analogous to the one above. However, corresponding to each zero of multiplicity m, there must be m linear equations, one stemming from point evaluation at the zero, and $m-1$ additional equations arising from point evaluation of the first $m-1$ derivatives at the zero. The details are not hard. We leave them to the reader.

We have proved the following theorem.

Theorem 14.2. *Suppose Ω is a bounded domain with C^∞ smooth boundary and suppose φ is a function in $C^\infty(b\Omega)$. Then the solution to the Dirichlet problem with boundary data φ exists and is in $C^\infty(\overline{\Omega})$.*

As in the simply connected case, this theorem, which solves the Dirichlet problem in C^∞, can be used to prove that the classical Dirichlet problem is solvable for continuous functions.

Note that by revisiting Theorem 11.3 in the multiply connected setting, and noting that the functions $\log|z-b_i|$ are harmonic on an open set containing the closure of the domain, we obtain the following theorem.

Theorem 14.3. *Suppose Ω is a bounded domain with smooth real analytic boundary and suppose φ is a function in $C^\omega(b\Omega)$. The solution to the Dirichlet problem with boundary data φ exists and extends to be harmonic on a neighborhood of $\overline{\Omega}$.*

15

The Bergman space

To begin, we suppose that Ω is merely a domain in the plane of *finite area* and we do not make any assumptions about the nature of the boundary. The *Bergman space*, denoted $H^2(\Omega)$, is the space of holomorphic functions on Ω that are square integrable on Ω with respect to area measure $dA = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$, i.e., h in $H^2(\Omega)$ are holomorphic functions such that $\iint_\Omega |h|^2 \, dA < \infty$. We may think of $H^2(\Omega)$ as being a subset of $L^2(\Omega)$ by adopting the standard convention that two functions that agree almost everywhere are the same function. Theorem 6.5 shows that $H^2(b\Omega)$ is a subset of $H^2(\Omega)$ when the domain is bounded and smooth.

We wish to define the orthogonal projection of $L^2(\Omega)$ onto its subspace of holomorphic functions similarly to the way we defined the Szegő projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$. To do this, we need to see that $H^2(\Omega)$ is a *closed* subspace of $L^2(\Omega)$. To prove this, we will use the fact that the value of a holomorphic function at the center of a disc is given by the average of the function over the disc. Thus, if $z_0 \in \Omega$ and $h \in H^2(\Omega)$, we may estimate the value of $h(z_0)$ by averaging h over the disc $D_r(z_0)$ where r is any radius that is less than the distance d from z_0 to the boundary of Ω. Using Hölder's inequality, we obtain

$$|h(z_0)| \leq \frac{1}{\pi r^2} \iint_{D_r(z_0)} |h| \, dA \leq \frac{1}{\sqrt{\pi}\, r} \left(\iint_{D_r(z_0)} |h|^2 \, dA \right)^{1/2},$$

and this last expression is less than or equal to $(\sqrt{\pi}\, r)^{-1} \|h\|$ where $\|h\|$ denotes the $L^2(\Omega)$ norm of h. By letting r tend to d, we obtain the improved estimate $|h(z_0)| \leq C\|h\|$ where $C = 1/(\sqrt{\pi}\, d)$. Thus, if h_j is a sequence of holomorphic functions on Ω that converge in $L^2(\Omega)$ to u, then h_j converges uniformly on compact subsets of Ω to a holomorphic function h. Since convergence in $L^2(\Omega)$ implies pointwise convergence almost everywhere of a subsequence, it follows that $u = h$ almost everywhere, i.e., that $u \in H^2(\Omega)$.

The space $L^2(\Omega)$ is a Hilbert space with inner product defined via $\langle u, v \rangle_\Omega = \iint_\Omega u\bar{v} \, dA$. Since $H^2(\Omega)$ is a closed subspace of a Hilbert space, it too is a Hilbert space.

The estimate $|h(a)| \leq C\|h\|$ that we proved above for $h \in H^2(\Omega)$ and $a \in \Omega$ shows that evaluating a function $h \in H^2(\Omega)$ at a point $a \in \Omega$ is a continuous linear functional on the Hilbert space $H^2(\Omega)$. Thus, the Riesz representation theorem implies that there is a function K_a in $H^2(\Omega)$ that represents this functional. The function K_a is called the *Bergman kernel function* and it is standard to write $K(z,a) = K_a(z)$. Because $K_a(b) = \langle K_a, K_b\rangle_\Omega$ and $K_b(a) = \langle K_b, K_a\rangle_\Omega$ it follows that $K(a,b) = \overline{K(b,a)}$. It also follows that $K(a,a) = \langle K_a, K_a\rangle_\Omega = \|K_a\|^2$ and this quantity must be positive because not every holomorphic function in $H^2(\Omega)$ vanishes at a (for example, $f(z) \equiv 1$ is such a nonvanishing function).

Since $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, we may consider the orthogonal projection B of $L^2(\Omega)$ onto $H^2(\Omega)$. This operator is called the *Bergman projection.* The Bergman kernel is the kernel for the Bergman projection in the sense that

$$(Bu)(a) = \langle Bu, K_a\rangle_\Omega = \langle u, K_a\rangle_\Omega = \iint_{z\in\Omega} K(a,z)u(z)\ dA.$$

The Bergman space is an important tool in the study of conformal mappings. That this should be so can be understood from the simple fact that if f is holomorphic, then $|f'|^2$ is equal to the *real* Jacobian determinant of f viewed as a mapping from $\mathbb{R}^2$ into itself. Thus, if $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded domains, then, without worrying about such things as convergence, the classical change of variables formula reads

$$\iint_{\Omega_2} \varphi\ dA = \iint_{\Omega_1} |f'|^2(\varphi \circ f)\ dA.$$

Taking $\varphi = |h|^2$ in this formula leads one to guess that the transformation $h \mapsto f'(h \circ f)$ is a norm preserving operator between the Bergman spaces associated to Ω_2 and Ω_1. The notation $f'(\varphi \circ f)$ stands for f' *times the quantity*, $\varphi \circ f$. This transformation is important, and so we will give it a name; let $\Lambda_1\varphi = f'(\varphi \circ f)$. In fact, this transformation is so important that it is worthwhile to be more careful in defining it. If φ is in $C_0^\infty(\Omega_2)$, then $\Lambda_1\varphi$ is in $C_0^\infty(\Omega_1)$ and the change of variables formula above is valid. Replacing φ by $|\varphi|^2$ in the change of variables formula shows that $\|\Lambda_1\varphi\|_{\Omega_1} = \|\varphi\|_{\Omega_2}$. Now $C_0^\infty(\Omega_2)$ is dense in $L^2(\Omega_2)$, and therefore, this estimate shows that Λ_1 extends uniquely as a bounded operator from $L^2(\Omega_2)$ to $L^2(\Omega_1)$. Furthermore, it is clear that Λ_1 preserves holomorphic functions. Let $F = f^{-1}$ and, for $\psi \in L^2(\Omega_1)$, define $\Lambda_2\psi = F'(\psi \circ F)$ analogous to the way we defined $\Lambda_1\phi$ above. It is easy to check, via the identity $f'(z) = 1/F'(f(z))$, that Λ_2 is the inverse to Λ_1. Thus, Λ_1 is an isometry between $L^2(\Omega_2)$ and $L^2(\Omega_1)$ that restricts