to be an isometry between $H^2(\Omega_2)$ and $H^2(\Omega_1)$. These facts, together with a little standard Hilbert space theory, can be used to prove another useful identity,

$$\langle \Lambda_1 u, v\rangle_{\Omega_1} = \langle u, \Lambda_2 v\rangle_{\Omega_2}, \tag{15.1}$$

that holds for all $u \in L^2(\Omega_2)$ and $v \in L^2(\Omega_1)$. Indeed, we proved a result identical to this when we proved identity (12.4). Recall that the polarization identity implies that an isometry also preserves the inner product, and so $\langle \Lambda_1 u_1, \Lambda_1 u_2\rangle_{\Omega_1} = \langle u_1, u_2\rangle_{\Omega_2}$. Hence, since $\Lambda_1\Lambda_2$ is the identity operator, we may write

$$\langle \Lambda_1 u, v\rangle_{\Omega_1} = \langle \Lambda_1 u, \Lambda_1(\Lambda_2 v)\rangle_{\Omega_1} = \langle u, \Lambda_2 v\rangle_{\Omega_2}$$

and the identity is proved.

In this chapter, we want to prove some facts about the Bergman projection and the Bergman kernel. To motivate why such a program might be fruitful, let us take a moment to show that the Bergman kernel function of a bounded simply connected domain is related to Riemann mapping functions. Let $f : \Omega \to D_1(0)$ be a Riemann mapping function of a bounded simply connected domain Ω onto the unit disc such that $f(a) = 0$ and $f'(a) > 0$. Let $F = f^{-1}$. The averaging property of holomorphic functions implies that $\langle H, 1\rangle_{D_1(0)} = \pi H(0)$ for all $H \in H^2(D_1(0))$. Now, if we apply the conjugate of the identity we proved above, using the functions $u \equiv 1$ and $v = h \in H^2(\Omega)$, we obtain

$$\langle h, f'\rangle_\Omega = \langle F'(h \circ F)\, ,\, 1\rangle_{D_1(0)} = \pi F'(0) h(F(0)) = c h(a)$$

where $c = \pi F'(0) = \pi/f'(a)$. It follows that the function $k(z) = c^{-1} f'(z)$ has the property that $\langle h, k\rangle_\Omega = h(a)$ for all $h \in H^2(\Omega)$. This property is shared by the Bergman kernel K_a. Hence, by the uniqueness clause in the Riesz representation theorem, it follows that $k(z) \equiv K_a(z)$, i.e., that $f'(z) = cK(z, a)$ where $c = \pi/f'(a)$. Letting $z = a$ in this formula reveals that $\pi K(a, a) = f'(a)^2$. Since $f'(a) > 0$, we may deduce that $f'(a) = \sqrt{\pi K(a, a)}$. Now we can write down the classical formula

$$f'(z) = CK(z, a)$$

where $C = \sqrt{\pi/K(a, a)}$. Thus, learning things about the Bergman kernel is going to yield information about conformal mappings.

Incidentally, we may use the formula for the Riemann map proved above to determine the Bergman kernel for the unit disc U. The averaging property for holomorphic functions implies that $\pi h(0) = \langle h, 1\rangle_U$ for every $h \in H^2(U)$. Hence, it follows that the Bergman kernel $K_U(z, w)$ for the disc satisfies $K_U(z, 0) \equiv \frac{1}{\pi}$ for $z \in U$. Fix a point $a \in U$. Let $f(z) = (z - a)/(1 - \bar{a}z)$. This map is the Riemann map of the unit disc onto the unit disc mapping a to the origin. Hence $f'(z) = C\,K_U(z,a)$. It follows that $K_U(z,a) = c(1-\bar{a}z)^{-2}$ where $c = (1-|a|^2)/C$. But since $K_U(z,0) = \frac{1}{\pi}$, we deduce that $c = 1/\pi$ and therefore that

$$K_U(z,w) = \frac{1}{\pi(1-\bar{w}z)^2}.$$

We now assume that Ω is a bounded domain in the plane *with C^∞ smooth boundary.* In this setting we wish to relate the Bergman projection to the Dirichlet problem, and hence, to the Szegő projection. To do this, we require the following lemma.

Lemma 15.1. *Suppose that Ω is a bounded domain with C^∞ smooth boundary. If φ is a function in $C^\infty(\overline{\Omega})$ that vanishes on $b\Omega$, then $\partial\varphi/\partial z$ is orthogonal to $H^2(\Omega)$.*

Proof. Suppose φ is as in the statement of the lemma. If h is in $A^\infty(\Omega)$, then a simple application of the complex Green's identity reveals that

$$\langle h, \partial\varphi/\partial z\rangle_\Omega = \iint_\Omega \frac{\partial}{\partial \bar{z}}(h\bar{\varphi})\left(-\frac{i}{2}d\bar{z}\wedge dz\right) = -\frac{i}{2}\int_{b\Omega} h\,\bar{\varphi}\;dz = 0.$$

Since we do not yet know that $A^\infty(\Omega)$ is dense in $H^2(\Omega)$, we cannot use a density argument to deduce the lemma from this simple computation. Instead, we must resort to the following machinations.

Consider a finite covering of $\overline{\Omega}$ by small discs and suppose that $\{\chi_j\}$ is a C^∞ partition of unity that is subordinate to the cover. If we can prove that $\partial(\chi_j\varphi)/\partial z$ is orthogonal to $H^2(\Omega)$ for each j, then it follows that $\partial\varphi/\partial z = \sum \partial(\chi_j\varphi)/\partial z$ is also orthogonal to $H^2(\Omega)$. Hence, we may reduce our problem to proving the lemma for $\varphi \in C^\infty(\overline{\Omega})$ that is supported on a small set of the form $D_\epsilon(z_0)\cap\Omega$ where z_0 is a fixed point in $b\Omega$. Of course, we continue to assume that $\varphi = 0$ on $b\Omega$. Let $\xi_0 = iT(z_0)$, i.e., let ξ_0 denote the complex number that represents the inward pointing unit normal vector to $b\Omega$ at z_0. If $\epsilon > 0$ is sufficiently small, the translation of the region $D_\epsilon(z_0)\cap\Omega$ by a small distance δ in the ξ_0 direction will be compactly contained in Ω. Given $h \in H^2(\Omega)$, for $z \in \Omega$, define $h_\delta(z) = h(z+\delta\xi_0)$ if $z+\delta\xi_0$ is in Ω and define $h_\delta(z) = 0$ otherwise. It is a standard fact in measure theory that h_δ tends to h in $L^2(\Omega)$ as $\delta \to 0$. If $h \in H^2(\Omega)$, then

$$\langle h, \partial\varphi/\partial z\rangle_\Omega = \lim_{\delta\to 0}\langle h_\delta, \partial\varphi/\partial z\rangle_{D_\epsilon(z_0)\cap\Omega}$$

and this last quantity is zero because h_δ is holomorphic on a neighborhood of the support of φ and we may use the complex Green's identity on $D_\epsilon(z_0)\cap\Omega$ as we did in the simple case above. □

We need to define an operator related to the Dirichlet problem. The *classical Green's operator* G is the solution operator to the following problem. Given $v \in C^\infty(\overline{\Omega})$, then Gv is equal to the function u satisfying

$$\begin{aligned} \Delta u &= v \qquad \text{on } \Omega \\ u &= 0 \qquad \text{on } b\Omega. \end{aligned} \tag{15.2}$$

Theorem 15.1. *Suppose that Ω is a bounded domain with C^∞ smooth boundary. Then the classical Green's operator for Ω is well defined and maps $C^\infty(\overline{\Omega})$ into itself.*

Proof. By Theorem 2.2, we can solve the equation $\partial u/\partial \bar{z} = v$ in the $C^\infty(\overline{\Omega})$ category, and, since $\overline{\partial u/\partial \bar{z}} = \partial \bar{u}/\partial z$, we can also solve the equation $\partial u/\partial z = v$ in the $C^\infty(\overline{\Omega})$ category. Let u_1 be a function in $C^\infty(\overline{\Omega})$ that satisfies $\partial u_1/\partial \bar{z} = \frac{1}{4}v$ and let u_2 be a function in $C^\infty(\overline{\Omega})$ that satisfies $\partial u_2/\partial z = u_1$. Because $(\partial/\partial \bar{z})(\partial/\partial z) = \frac{1}{4}\Delta$, it follows that $\Delta u_2 = v$. Finally, let u_3 be the harmonic function on Ω that has u_2 as its boundary values (u_3 is in $C^\infty(\overline{\Omega})$ by Theorem 14.2). Now $u = u_2 - u_3$ solves problem (15.2).

This solution u is the unique solution to problem (15.2) in the following strong sense. Assume that U is a function that is merely continuous on $\overline{\Omega}$ and C^∞ smooth on Ω. Assume further that $\Delta U = v$ on Ω and $U = 0$ on $b\Omega$. Then $u - U$ is a harmonic function on Ω that is continuous on $\overline{\Omega}$ and that vanishes on $b\Omega$, and so the maximum principle for harmonic functions implies that $u - U \equiv 0$. Hence, the Green's operator is well defined and the proof is finished. □

The next theorem shows how the Bergman projection is related to the Dirichlet problem via the Green's operator.

Theorem 15.2. *Suppose that Ω is a bounded domain with C^∞ smooth boundary. For $u \in C^\infty(\overline{\Omega})$, the Bergman projection Bu of u is given by*

$$Bu = u - 4\frac{\partial}{\partial z} G \frac{\partial u}{\partial \bar{z}}.$$

It follows that B maps $C^\infty(\overline{\Omega})$ into itself. It also follows that the Bergman kernel $K_a(z) = K(z,a)$ is in $A^\infty(\Omega)$ as a function of z for each fixed $a \in \Omega$.

Proof. Let $\varphi = 4G\frac{\partial u}{\partial \bar{z}}$. Since φ vanishes on $b\Omega$, it follows that $(\partial\varphi/\partial z)$ is orthogonal to $H^2(\Omega)$. Hence $B(\partial\varphi/\partial z) = 0$ and it follows that $Bu = B\left(u - \frac{\partial}{\partial z}\varphi\right)$. But $u - \frac{\partial}{\partial z}\varphi$ is holomorphic because the computation,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z} 4G \frac{\partial u}{\partial \bar{z}}\right) = \frac{1}{4}\Delta\left(4G\frac{\partial u}{\partial \bar{z}}\right) = \frac{\partial u}{\partial \bar{z}},$$

shows that $\frac{\partial}{\partial \bar{z}}(u - \frac{\partial}{\partial z}\varphi)$ is zero, i.e., that $u - \frac{\partial}{\partial z}\varphi$ satisfies the Cauchy-Riemann equations. Hence, $Bu = B(u - \frac{\partial}{\partial z}\varphi) = u - \frac{\partial}{\partial z}\varphi$, and this is what we wanted to see.

To finish the proof, we need to express the Bergman kernel as the Bergman projection of a smooth function. The easiest way to do this is to let φ_a be a function in $C_0^\infty(\Omega)$ that is radially symmetric about the point a such that $\iint_\Omega \varphi_a \, dA = 1$. Because the value of a holomorphic function $h \in H^2(\Omega)$ is equal to $(2\pi)^{-1} \int_0^{2\pi} h(a + re^{i\theta})\, d\theta$ when r is less than the distance from a to $b\Omega$, it can be verified by integrating in polar coordinates centered at a that $\langle h, \varphi_a \rangle_\Omega = h(a)$ for all $h \in H^2(\Omega)$. Hence, $K_a = B\varphi_a$ and the smoothness of K_a follows from the fact that the Bergman projection preserves $C^\infty(\overline{\Omega})$ because the Green's operator does. □

The fact that B preserves the space $C^\infty(\overline{\Omega})$ has an important corollary.

Corollary 15.1. *If Ω is a bounded domain with C^∞ smooth boundary, then $A^\infty(\Omega)$ is dense in $H^2(\Omega)$.*

Proof. Given $h \in H^2(\Omega)$, let u_j be a sequence in $C^\infty(\overline{\Omega})$ that converges to h in $L^2(\Omega)$. Now Bu_j is a sequence in $A^\infty(\Omega)$ that converges in $L^2(\Omega)$ to $Bh = h$. □

The formula in Theorem 15.2 is called *Spencer's formula*. It is worth remarking that the proof of Spencer's formula contained a theorem about solving a $\bar{\partial}$-problem. The problem is, given $v \in C^\infty(\overline{\Omega})$, find $u \in C^\infty(\overline{\Omega})$ such that $\partial u/\partial \bar{z} = v$ with u orthogonal to the Bergman space. Scrutiny of the proof of Spencer's formula reveals that a solution to this problem is given by $u = 4\frac{\partial}{\partial z}Gv$. Uniqueness is easy because if u_1 and u_2 both solve the problem, then $u_1 - u_2$ would be a holomorphic function that is orthogonal to holomorphic functions, and hence $u_1 - u_2 \equiv 0$.

It is useful to know that $A^\infty(\Omega)$ is dense in the Bergman space. Next, we seek a nice dense subspace of the orthogonal complement $H^2(\Omega)^\perp$ of $H^2(\Omega)$ in $L^2(\Omega)$. The next few results do not require us to assume that the boundary of our domain is smooth, and therefore we drop this assumption. We will only assume that the domain Ω under study is bounded. A function g in $L^2(\Omega)$ is called a *weak solution to the Cauchy-Riemann equations* (or a distributional solution to the Cauchy-Riemann equations) if

$$\iint_\Omega g \, \frac{\partial \varphi}{\partial \bar{z}} \, dA = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Since $\overline{\partial\varphi/\partial z} = \partial\bar{\varphi}/\partial\bar{z}$, an equivalent way of stating this condition is to say

$$\langle g, \frac{\partial \varphi}{\partial z} \rangle_\Omega = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. To make sense of this definition, consider what it implies about a function g with continuous first partial derivatives. In this setting, we may integrate by parts via the complex Green's identity to obtain

$$\iint_\Omega g \; \frac{\partial \varphi}{\partial \bar{z}} \; dA = - \iint_\Omega \frac{\partial g}{\partial \bar{z}} \; \varphi \; dA.$$

If this integral vanishes for all smooth compactly supported φ, then $\frac{\partial g}{\partial \bar{z}}$ must be zero and therefore g satisfies the Cauchy-Riemann equations. Hence, a weak solution to the Cauchy-Riemann equations that has continuous first partial derivatives is a strong solution.

We will need the following classical result, known as *Weyl's lemma*, that says that a locally integrable function that is a weak solution to the Cauchy-Riemann equations is holomorphic (in the sense that there exists a holomorphic function with which it agrees almost everywhere).

Lemma 15.2. *Suppose Ω is a bounded domain and suppose $g \in L^2(\Omega)$ is a weak solution to the Cauchy-Riemann equations. Then $g \in H^2(\Omega)$. Consequently, the set of functions $\mathcal{F}$ of the form $\frac{\partial \varphi}{\partial z}$, where $\varphi \in C_0^\infty(\Omega)$, is a dense subspace of $H^2(\Omega)^\perp$.*

Proof. Let θ denote a function in $C_0^\infty(D_1(0))$ that is radially symmetric such that $\iint_{D_1(0)} \theta \; dA = 1$, and let $\theta_\epsilon(z) = \epsilon^{-2}\theta(z/\epsilon)$. Consider the convolution $g_\epsilon = \theta_\epsilon * g$ which is well defined on the set Ω_ϵ of points in Ω that are greater than a distance of ϵ from the boundary. First, we show that g_ϵ is holomorphic on Ω_ϵ. Indeed, it is a standard exercise in real analysis to see that g_ϵ is C^∞ on Ω_ϵ. Furthermore, on Ω_ϵ,

$$\frac{\partial}{\partial \bar{z}} g_\epsilon = \left(\frac{\partial \theta_\epsilon}{\partial \bar{z}} \right) * g = \iint_{w \in \Omega} \frac{\partial}{\partial \bar{z}} [\theta_\epsilon(z-w)] \; g(w) dA.$$

But $\frac{\partial}{\partial \bar{z}}[\theta_\epsilon(z-w)] = -\frac{\partial}{\partial \bar{w}}[\theta_\epsilon(z-w)]$. Therefore, the last integral above is equal to

$$- \iint_{w \in \Omega} \frac{\partial}{\partial \bar{w}} [\theta_\epsilon(z-w)] \; g(w) dA,$$

and this quantity is zero by the definition of weak solutions to the Cauchy-Riemann equations. Hence g_ϵ is holomorphic on Ω_ϵ. Let ϵ_j be a sequence of positive numbers that tend to zero and let $g_j = g_{\epsilon_j}$ on Ω_{ϵ_j} and $g_j = 0$ on $\Omega - \Omega_{\epsilon_j}$. It is another standard exercise in real analysis to see that g_j converges in $L^2(\Omega)$ to g. As in the proof that $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, we see that g is locally the L^2 limit of holomorphic functions and that, therefore, g must be equal almost everywhere to a function holomorphic on Ω.

Finally, we must check the density statement. If $\mathcal{F}$ is not dense in $H^2(\Omega)^\perp$, then there would exist a nonzero element g of $H^2(\Omega)^\perp$ that is orthogonal to $\mathcal{F}$. But this orthogonality condition is equivalent to saying that g is a weak solution to the Cauchy-Riemann equations. Thus, by the lemma, $g \in H^2(\Omega)$, and we must conclude that $g \equiv 0$, contrary to hypothesis. The proof is finished. □

16

Proper holomorphic mappings and the Bergman projection

We showed in the last chapter that the Bergman kernel is related to conformal mappings. We now wish to study a more general class of holomorphic mappings between domains than conformal mappings. We will show that the Bergman projection and kernel are also useful in the study of these mappings.

A continuous mapping is called *proper* if the inverse image of any compact set is compact. It is easy to check that biholomorphic mappings are proper. Finite Blaschke products are examples of proper holomorphic maps of the unit disc into itself, and it is a standard exercise in complex analysis to show that these maps constitute all possible proper holomorphic self maps of the unit disc. The Ahlfors map studied in Chapter 13 is another example of a proper holomorphic mapping. Indeed, if $f : \Omega_1 \to \Omega_2$ is a holomorphic mapping between bounded domains that extends continuously up to the boundary, then it is an easy exercise to check that the condition that f be proper is equivalent to the condition that $f(b\Omega_1) \subset b\Omega_2$. Proper holomorphic mappings between domains share many of the nice qualities held by conformal mappings.

Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map between bounded domains. We wish to define the operator Λ_1 for f as we did for biholomorphic maps in Chapter 15. Given a function ϕ on Ω_2, we define $\Lambda_1\phi = f'(\phi \circ f)$, which is defined on Ω_1. We wish to show that Λ_1 maps $L^2(\Omega_2)$ into $L^2(\Omega_1)$. As in the biholomorphic case, a key element of the proof will be that $|f'|^2$ is equal to the *real* Jacobian determinant of f, viewed as a mapping of $\mathbb{R}^2$ into itself. However, before we can use the classical change of variables formula, we will need to know some elementary properties of proper holomorphic maps. We need to know that f maps Ω_1 *onto* Ω_2, and that there is a positive integer m, known as the *multiplicity of* f, such that (counting multiplicities), f is an m-to-one map of Ω_1 onto Ω_2.

Since clearly f cannot be constant, the open mapping theorem says that $f(\Omega_1)$ is an open subset of Ω_2. We claim that $f(\Omega_1)$ is also a relatively closed subset of Ω_2, and that therefore, since Ω_2 is connected,

$f(\Omega_1) = \Omega_2$. That $f(\Omega_1)$ is closed in Ω_2 follows from the definition of a proper map. Indeed, if w_i is a sequence in $f(\Omega_1)$ converging to a point $w_0 \in \Omega_2$, then the set $K = \cup_{i=0}^{\infty}\{w_i\}$ is compact. Hence, $f^{-1}(K)$ is compact. We may choose a sequence z_i of points in Ω_1 such that $f(z_i) = w_i$. Since $f^{-1}(K)$ is compact, there is a subsequence of z_i converging to a point $z_0 \in f^{-1}(K) \subset \Omega_1$. Now the continuity of f implies that $f(z_0) = w_0$, and hence, that $w_0 \in f(\Omega_1)$.

We have shown that f maps Ω_1 *onto* Ω_2. Let $w_0 \in \Omega_2$. Since f is not constant, the set $f^{-1}(w_0)$ is discrete in Ω_1. Since f is proper, this set is also compact, and it must therefore be a finite set in Ω_1. Write $f^{-1}(w_0) = \{a_1, a_2, \ldots, a_n\}$ where the a_i are distinct and let m_i denote the multiplicity of the zero of $f(z) - w_0$ at $z = a_i$. Let $m = \sum_{i=1}^{n} m_i$. We now claim that this number does not depend on the choice of w_0. To see this, we use the fact proved in most elementary books on complex variables (see Ahlfors [Ah, p. 132]) that, for sufficiently small $\epsilon > 0$, there are neighborhoods U_i of each a_i such that $f : U_i \to D_\epsilon(w_0)$ is an m_i-to-one covering map of $U_i - \{a_i\}$ onto $D_\epsilon(w_0) - \{w_0\}$. In fact, the map is given as a composition of the map $w_0 + (z - a_i)^{m_i}$ with a one-to-one holomorphic map. By shrinking ϵ if necessary, we may assume that the U_i are disjoint. We will now prove there is a $\rho < \epsilon$ such that if $|w - w_0| < \rho$, then $f^{-1}(w) \subset \cup_{i=1}^{n} U_i$. If this were not true, there would exist a sequence w_i tending to w_0 and points $z_i \in \Omega_1 - (\cup_{i=1}^{n} U_i)$ such that $f(z_i) = w_i$. By the same argument we used to show that $f(\Omega_1)$ is closed in Ω_2, it would follow that there is a convergent subsequence of z_i converging to a point $z_0 \in \Omega_1$. By continuity, $f(z_0) = w_0$, and therefore, $z_0 = a_k$ for some k. But this forces us to conclude that some of the z_i lie in U_k, contrary to assumption. Hence, the existence of ρ is assured, and it follows that if $|w - w_0| < \rho$ and $w \neq w_0$, then $f^{-1}(w)$ consists of exactly m *distinct* points, m_i of them falling in U_i. Hence, the number m associated to w is the same as that associated to w_0, and we have shown that $m(w)$ is a locally constant function of w on Ω_2. A locally constant function on a connected set is constant.

In fact, the proof above gives more than the existence of m. Let $\mathcal{L}$ denote the set of points $z \in \Omega_1$ such that $f'(z) = 0$. This set is known as the *branch locus* of f and the set $f(\mathcal{L})$ is called the *image of the branch locus*. The proof above shows that $f(\mathcal{L})$ is a discrete subset of Ω_2 and that for each w not in $f(\mathcal{L})$, the set $f^{-1}(w)$ consists of exactly m distinct points. Furthermore, if $w \in f(\mathcal{L})$, the set $f^{-1}(w)$ consists of strictly fewer than m points. Let $V_2 = f(\mathcal{L})$ and let $V_1 = f^{-1}(V_2)$. The sets V_i are discrete subsets of Ω_i, $i = 1, 2$, and we have shown that f is an m-to-one (unbranched) covering map of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$. Note that discrete subsets are sets of measure zero with respect to Lebesgue area measure. Thus, integrals over Ω_i are equal to integrals over $\Omega_i - V_i$.

We are now prepared to study the operator Λ_1. Let $\varphi \in C_0^\infty(\Omega_2 - V_2)$. Since V_2 is discrete, it is easy to check that such functions are dense in $L^2(\Omega_2)$. Because f is an m-to-one unbranched covering map of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$, we may apply the classical change of variables formula to write

$$\begin{aligned}\iint_{\Omega_1} |f'|^2 |\phi \circ f|^2 \, dA &= \iint_{\Omega_1 - V_1} |f'|^2 |\phi \circ f|^2 \, dA \\ &= m \iint_{\Omega_2 - V_2} |\phi|^2 \, dA = m \iint_{\Omega_2} |\phi|^2 \, dA.\end{aligned}$$

This formula is easy to understand when φ is supported in a very small disc. Hence, we can verify the formula by using a partition of unity $\{\chi_j\}$ and by summing the integrals $\iint_{\Omega_2} \chi_j |\varphi|^2 \, dA$. Our calculation shows that $\|\Lambda_1 \varphi\|^2 = m\|\varphi\|^2$. By density, Λ_1 extends uniquely to all of $L^2(\Omega_2)$ as a bounded operator to $L^2(\Omega_1)$ with operator norm $\sqrt{m}$. Since convergence in L^2 implies almost everywhere convergence of a subsequence, it follows that the extension of Λ_1 to $L^2(\Omega_2)$ can be expressed via the formula $\Lambda_1 \phi = f'(\phi \circ f)$, and so our original definition of Λ_1 agrees with the operator constructed by extension.

Let a subscript i on a Bergman projection indicate that it is associated to the domain Ω_i. Lemma 15.2 will allow us to give a simple proof of the following transformation formula for the Bergman projections under proper holomorphic mappings.

Theorem 16.1. *Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map between bounded domains. Then, the operator Λ_1 commutes with the Bergman projection in the sense that $B_1 \Lambda_1 = \Lambda_1 B_2$, i.e.,*

$$B_1 \left(f'(\varphi \circ f) \right) = f' \left((B_2 \varphi) \circ f \right)$$

for all $\varphi \in L^2(\Omega_2)$.

Proof. The formula is clearly true when $\varphi = h \in H^2(\Omega_2)$. In this case, $B_2 h = h$. Now $\Lambda_1 h$ is holomorphic and in the Bergman space of Ω_1 because Λ_1 is a bounded operator. Hence, $B_1 \Lambda_1 h = \Lambda_1 h$. Thus $B_1 \Lambda_1 h = \Lambda_1 h = \Lambda_1 B_2 h$.

Because the formula is true on $H^2(\Omega_2)$, and because Λ_1 and the Bergman projections are bounded *linear* operators, we may reduce our task to showing that the transformation formula holds for φ in $\mathcal{F}_2$, the dense subspace of $H^2(\Omega_2)^\perp$ mentioned in Lemma 15.2. If ψ has compact support in Ω_2, then $\psi \circ f$ has compact support in Ω_1. Furthermore, the complex chain rule yields $(\partial/\partial z)(\psi \circ f) = f'[(\partial \psi/\partial z) \circ f)]$. Hence, if $\varphi \in \mathcal{F}_2$, then $\Lambda_1 \varphi \in \mathcal{F}_1$, the dense subspace of $H^2(\Omega_1)^\perp$ of Lemma 15.2. Therefore, for such a φ, it follows that $B_2 \varphi = 0$, and $B_1 \Lambda_1 \varphi = 0$, and so

$B_1\Lambda_1\varphi = \Lambda_1 B_2\varphi$ because both of these expressions are zero. The proof of the transformation formula is complete. □

Next, we prove a general theorem about the boundary behavior of proper maps using the transformation formula for the Bergman projections.

Theorem 16.2. *Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map between bounded domains with C^∞ smooth boundaries. Then $f \in C^\infty(\overline{\Omega}_1)$ and f' is nonvanishing on $b\Omega_1$. It follows that f maps the boundary of Ω_1 into the boundary of Ω_2.*

Proof. Because of Lemma 12.1, we may assume that Ω_1 and Ω_2 have real analytic boundaries. Lemma 11.1 implies that there is a function $\Phi \in C^\infty(\overline{\Omega}_2)$ such that $\Phi = 0$ on $b\Omega_2$ and such that $1 - (\partial\Phi/\partial z)$ has compact support in Ω_2. Remember that the conjugate of $\partial u/\partial \bar{z}$ is $\partial \bar{u}/\partial z$. Since Φ vanishes on $b\Omega_2$, $\partial\Phi/\partial z$ is orthogonal to holomorphic functions on Ω_2 by Lemma 15.1. Now, the function ψ given by

$$\psi = 1 - \frac{\partial\Phi}{\partial z}$$

is such that $B_2\psi \equiv 1$ and $\psi \in C_0^\infty(\Omega_2)$. Let 1 denote the function that is identically one on Ω_2. The transformation formula for the Bergman projections yields that

$$f' = f'(1 \circ f) = f'\left((B_2\psi)\circ f\right) = B_1\left(f'(\psi \circ f)\right).$$

Since f is proper and $\psi \in C_0^\infty(\Omega_2)$, it follows that $f'(\psi \circ f)$ is in $C_0^\infty(\Omega_1)$. Hence, f' is equal to the projection of a function in $C^\infty(\overline{\Omega}_1)$, and by Theorem 15.2, $f' \in C^\infty(\overline{\Omega}_1)$; so $f \in C^\infty(\overline{\Omega}_1)$. We have mentioned before that a proper map that extends continuously to the boundary must map the boundary to the boundary; so $f(b\Omega_1) \subset b\Omega_2$. Now the Schwarz Reflection Principle implies that f extends to be holomorphic in a neighborhood of $\overline{\Omega}_1$.

Finally, we must show that f' is nonvanishing on $b\Omega_1$. Let $z_0 \in b\Omega_1$. We may make local holomorphic changes of variables near z_0 and near $f(z_0)$ so that we are dealing with a map F that is holomorphic near the origin that maps the real line into itself, and such that $F(0) = 0$. We may further assume that F maps the upper half plane near the origin into the upper half plane. In the changed variables, F can be written $F(z) = z^m h(z)$ where h is holomorphic near zero, $h(0) \neq 0$, and $m \geq 1$. Note that, because F maps $\mathbb{R}$ into $\mathbb{R}$, $h(z)$ is real valued on the real axis. Since F maps the upper half plane into itself near the origin, it follows that $h(0) > 0$. Let $H(z)$ be a holomorphic m-th root of $h(z)$ such that $H(0)$ is a positive real number. Note that

$zH(z)$ is one-to-one near the origin and that $zH(z)$ maps $\mathbb{R}$ into $\mathbb{R}$. Furthermore, $zH(z)$ maps the upper half plane near the origin into the upper half plane because $H(0) > 0$. Hence, there is a small disc $D_\epsilon(0)$ such that $\{z : \text{Im } z > 0\} \cap D_\epsilon(0)$ is contained in the image under $zH(z)$ of a neighborhood of the origin. Now, if $m > 1$, it is easy to see that $F(z) = [zH(z)]^m$ maps the upper half plane near the origin onto a full neighborhood of the origin. This contradiction forces us to conclude that $m = 1$, and consequently $f'(z_0) \neq 0$. The proof is complete.

Alternatively, we could have used a Harnack inequality argument like the one used in the proof of Theorem 8.2 to prove that $f'(z_0) \neq 0$. $\square$

We now wish to show that the Bergman kernel transforms under proper holomorphic mappings. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains. Again, we drop any assumptions about the smoothness of the boundaries of Ω_i. Let $K_i(z, w)$ denote the Bergman kernel function associated to Ω_i, $i = 1, 2$. We have shown that there are discrete sets $V_i \subset \Omega_i$, $i = 1, 2$, such that f is an m-to-one covering map of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$. If $D_\epsilon(w_0) \subset \Omega_2 - V_2$, then there are holomorphic functions $F_k(w)$, $k = 1, 2, \ldots, m$, mapping $D_\epsilon(w_0)$ into $\Omega_1 - V_1$ such that $f(F_k(w)) = w$ there. The functions F_k are called the local inverses to f.

We need to construct an operator like the Λ_2 operator we used in the discussion of biholomorphic maps. Given $\varphi \in C^\infty(\Omega_1)$, we define $\Lambda_2\varphi$ on $\Omega_2 - V_2$ by setting

$$\Lambda_2\varphi = \sum_{k=1}^{m} F_k'(\varphi \circ F_k)$$

on small discs contained in $\Omega_2 - V_2$ as above. To see that this gives rise to a well defined C^∞ function on $\Omega_2 - V_2$, notice that on a small disc, the definition is clearly C^∞ and is independent of the order in which we choose to label the local inverses. Furthermore, the functions defined on two small discs agree on the intersection of the discs if they overlap. Hence, $\Lambda_2\varphi$ is a globally well defined C^∞ function on $\Omega_2 - V_2$. We wish to define this operator on $L^2(\Omega_1)$. Suppose that $\varphi \in C^\infty(\overline{\Omega}_1)$. To see that $\Lambda_2\varphi$ is square integrable, we will use the simple inequality $|\sum_{k=1}^m c_k|^2 \leq m\sum_{k=1}^m |c_k|^2$ which can be deduced by applying the Schwarz inequality for the inner product of two vectors in $\mathbb{C}^m$ to the vectors $(c_1, \ldots, c_m)$ and $(1, \ldots, 1)$. Now

$$\int_{\Omega_2 - V_2} \left| \sum_{k=1}^{m} F_k'(\varphi \circ F_k) \right|^2 dA \leq m \int_{\Omega_2 - V_2} \sum_{k=1}^{m} |F_k'(\varphi \circ F_k)|^2 \, dA,$$

and, using the facts that $|F_k'|^2$ represents a *real* Jacobian and that f is an m-sheeted covering map of $\Omega_1 - V_1$ onto $\Omega_2 - V_2$, it follows that the last quantity is equal to

$$m \int_{\Omega_1 - V_1} |\varphi|^2 \, dA.$$

As before, the calculation above is more transparent if φ is assumed to have very small support in $\Omega_1 - V_1$. The general case can be deduced by using a partition of unity. Since V_2 is a set of measure zero in Ω_2, it follows that $\Lambda_2\varphi$ can be viewed as an element of $L^2(\Omega_2)$, and our calculation shows that $\|\Lambda_2\varphi\|_{L^2(\Omega_2)} \leq \sqrt{m}\|\varphi\|_{L^2(\Omega_1)}$. Hence, since $C^\infty(\overline{\Omega}_1)$ is dense in $L^2(\Omega_1)$, it follows from this estimate that Λ_2 has a unique extension as a bounded operator from $L^2(\Omega_1)$ to $L^2(\Omega_2)$.

We next claim that Λ_2 maps $H^2(\Omega_1)$ into $H^2(\Omega_2)$. Given $h \in H^2(\Omega_1)$, we know that $\Lambda_2 h$ is holomorphic on $\Omega_2 - V_2$ and that $\Lambda_2 h$ is square integrable over $\Omega_2 - V_2$. The following generalization of the Riemann removable singularity theorem will show that $\Lambda_2 h \in H^2(\Omega_2)$.

Theorem 16.3. *Suppose that h is holomorphic on $\Omega - V$ where Ω is a bounded domain and V is a discrete subset of Ω. If $\int_{\Omega - V} |h|^2 \, dA < \infty$, then h has removable singularities at each point in V.*

Proof. The theorem is purely local, so we might as well assume that Ω is the unit disc and that $V = \{0\}$. The monomials $\{z^n\}_{n=-\infty}^{\infty}$ are orthogonal on each annulus $A_\epsilon^\rho = \{z : \epsilon < |z| < \rho\}$ where $0 < \epsilon < \rho < 1$. Furthermore, h has a Laurent expansion $\sum_{n=-\infty}^{\infty} a_n z^n$ that converges uniformly on each of these annuli. Hence

$$\iint_{A_\epsilon^\rho} |h|^2 \, dA = \sum_{n=-\infty}^{\infty} |a_n|^2 \iint_{A_\epsilon^\rho} |z^n|^2 \, dA$$

is bounded independent of ϵ. But a simple computation in polar coordinates shows that $\iint_{A_\epsilon^\rho} |z^n|^2 \, dA$ tends to infinity as ϵ tends to zero when $n < 0$. Hence, we must conclude that $a_n = 0$ if $n < 0$, i.e., that h has a removable singularity at the origin. The proof is finished. □

Next, we show that Λ_1 and Λ_2 are adjoint operators.

Theorem 16.4. *If $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains, then*

$$\langle \Lambda_1 u, v \rangle_{\Omega_1} = \langle u, \Lambda_2 v \rangle_{\Omega_2}$$

for all $u \in L^2(\Omega_2)$ and $v \in L^2(\Omega_1)$.

Proof. This is easy to prove if u is supported in a small disc $D_\epsilon(w_0) \subset \Omega_2 - V_2$ such that the local inverse maps $F_k(w)$ define biholomorphic maps of $D_\epsilon(w_0)$ onto disjoint domains W_k in $\Omega_1 - V_1$. In this case, we can use the analogous result for biholomorphic maps to see that

$$\begin{aligned}\langle \Lambda_1 u, v\rangle_{\Omega_1} &= \iint_{\Omega_1} f'(u\circ f)\,\overline{v}\,dA = \sum_{k=1}^{m} \iint_{W_k} f'(u\circ f)\,\overline{v}\,dA \\ &= \sum_{k=1}^{m} \iint_{D_\epsilon(w_0)} u\,\overline{F_k'(v\circ F_k)}\,dA = \iint_{D_\epsilon(w_0)} u\,\overline{\Lambda_2 v}\,dA \\ &= \langle u, \Lambda_2 v\rangle_{\Omega_2}.\end{aligned}$$

The theorem now follows because the linear span of the set of functions in $L^2(\Omega_2)$ that are supported in small discs of this type forms a dense subset of $L^2(\Omega_2)$, and a standard limiting argument can be used. □

Theorem 16.4 will allow us to deduce the transformation formula for the Bergman kernel functions under proper holomorphic mappings.

Theorem 16.5. *If $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains, then the Bergman kernels $K_j(z,w)$ associated to Ω_j, $j = 1, 2$, transform according to*

$$f'(z)K_2(f(z), w) = \sum_{k=1}^{m} K_1(z, F_k(w))\overline{F_k'(w)}$$

where the multiplicity of the mapping is m and the functions $F_k(w)$ denote the local inverses to f.

Notice that when f is a biholomorphic mapping, the transformation rule in the statement of the theorem reduces to

$$f'(z)K_2(f(z), w) = K_1(z, F(w))\overline{F'(w)}$$

where $F = f^{-1}$. We may let $w = f(\zeta)$ in this formula and use the fact that $f'(\zeta) = 1/F'(f(\zeta))$ to write the last formula in the equivalent form

$$f'(z)K_2(f(z), f(\zeta))\overline{f'(\zeta)} = K_1(z, \zeta). \tag{16.1}$$

This result is known as the transformation formula for the Bergman kernel under biholomorphic mappings, and Theorem 16.5 is its generalization to the case of proper holomorphic mappings.

Proof of Theorem 16.5. Fix a point w in $\Omega_2 - V_2$ and let $g_1(z) = K_2(z, w)$. Notice that if $h \in H^2(\Omega_1)$, then

$$\langle h, \Lambda_1 g_1\rangle_{\Omega_1} = \langle \Lambda_2 h, g_1\rangle_{\Omega_2} = (\Lambda_2 h)(w)$$

because of the reproducing property of the Bergman kernel on Ω_2. Next, define $G_2(z) = \sum_{k=1}^{m} K_1(z, F_k(w))\overline{F_k'(w)}$. Now

$$\begin{aligned}\langle h, G_2\rangle_{\Omega_1} &= \sum_{k=1}^{m} F_k'(w)\langle h, K_1(\cdot, F_k(w))\rangle_{\Omega_1} \\ &= \sum_{k=1}^{m} F_k'(w) h(F_k(w)) = (\Lambda_2 h)(w)\end{aligned}$$

by the reproducing property of the Bergman kernel on Ω_1. We have shown that $\langle h, \Lambda_1 g_1\rangle_{\Omega_1} = \langle h, G_2\rangle_{\Omega_1}$ for any $h \in H^2(\Omega_1)$. It follows that $\Lambda_1 g_1 = G_2$, and this is precisely the transformation formula we are seeking to prove. Hence, the formula holds when $w \in \Omega_2 - V_2$. The formula is true for w in the discrete set V_2 in the sense that those points are removable singularities of the anti-holomorphic function on the right hand side of the formula, and therefore, when the function on the right hand side is given the proper value at these points, the transformation formula remains valid by continuity. □

In defining the operator Λ_2, we showed that the operator norm of Λ_2 was less than or equal to $\sqrt{m}$, where m is the multiplicity of the proper map f. Actually, the operator norm of Λ_2 is *equal* to $\sqrt{m}$, which is the same as the operator norm of Λ_1. We will now prove this. Observe that, since $f(F_k(w)) = w$, it follows that $F_k'(w)f'(F_k(w)) = 1$, and therefore, that $\Lambda_2\Lambda_1 u = mu$ for all u in $L^2(\Omega_2)$. This fact allows us to deduce that the range $\mathcal{R}$ of Λ_1 is closed. Indeed, if $\Lambda_1 u_i$ tends to U, then $u_i = \frac{1}{m}\Lambda_2\Lambda_1 u_i$ tends to $\frac{1}{m}\Lambda_2 U$, and this shows that U is equal to $\Lambda_1(\frac{1}{m}\Lambda_2 U)$. Since $\mathcal{R}$ is closed, $L^2(\Omega_1)$ has an orthogonal decomposition as $\mathcal{R} \oplus \mathcal{R}^\perp$. The adjoint law shows that $\mathcal{R}^\perp$ is equal to the kernel of the operator Λ_2. Hence, given $U \in L^2(\Omega_1)$, there exists a $u \in L^2(\Omega_2)$ and a $V \in L^2(\Omega_1)$ with $\Lambda_2 V \equiv 0$ such that $U = f'(u \circ f) + V$. This decomposition allows us to determine the operator norm of Λ_2. Indeed, the decomposition shows that $\Lambda_2 U = \Lambda_2\Lambda_1 u = mu$. But $\|U\|^2 = \|\Lambda_1 u\|^2 + \|V\|^2$ and we know from above that $\|\Lambda_1 u\| = \sqrt{m}\|u\|$. Hence, $\|\Lambda_2 U\| = m\|u\| = \sqrt{m}\|\Lambda_1 u\| \leq \sqrt{m}\|U\|$. It is clear that equality holds in this last inequality whenever $V = 0$, and it follows that the operator norm of Λ_2 is $\sqrt{m}$.

The same reasoning shows that the image of $H^2(\Omega_2)$ under Λ_1 is a closed subspace of $H^2(\Omega_1)$, and that therefore any $G \in H^2(\Omega_1)$ can be decomposed as $G = f'(g \circ f) + H$ where $g \in H^2(\Omega_2)$ and $H \in H^2(\Omega_1)$ with $\Lambda_2 H \equiv 0$. Furthermore, the operator norm of Λ_2 as an operator from $H^2(\Omega_1)$ to $H^2(\Omega_2)$ is equal to $\sqrt{m}$.

In case f is a biholomorphic map, the operator Λ_1 is an isometry and Λ_2 is its inverse. In case f is a proper holomorphic mapping of multiplicity $m > 1$, the operator $m^{-1/2}\Lambda_1$ is isometric and $m^{-1/2}\Lambda_2$ is an isometric right inverse. If f does not have a holomorphic inverse, Λ_1 does not map onto $H^2(\Omega_1)$.

17

The Solid Cauchy transform

We now wish to express the Bergman projection in terms of the Szegő projection. Suppose Ω is a bounded domain with C^∞ smooth boundary and suppose $v \in C^\infty(\overline{\Omega})$. We want to compute Bv, the Bergman projection of v. Define

$$(\Lambda v)(z) = \frac{1}{2\pi i} \iint_{w \in \Omega} \frac{v(w)}{\bar{w} - \bar{z}} \, dw \wedge d\bar{w}$$

for $z \in \Omega$. The operator Λ maps $C^\infty(\overline{\Omega})$ into itself and $(\partial \Lambda v / \partial z) = v$ on Ω (see the remark after Theorem 2.2). Let $\mathcal{E}$ denote the Poisson extension operator mapping a function $u \in C^\infty(\overline{\Omega})$ to the harmonic function on Ω that has the same boundary values as u. In this book, we have expressed $\mathcal{E}$ in terms of the Szegő projection (see Theorem 10.1 and Chapter 14).

We now claim that

$$Bv = \frac{\partial}{\partial z}(\mathcal{E}\Lambda v). \tag{17.1}$$

To see this, suppose that $h \in A^\infty(\Omega)$. Writing $u = \Lambda v$, we may compute as follows.

$$\begin{aligned}
\iint_\Omega v(z)\, \overline{h(z)} \, dx \wedge dy &= -\frac{1}{2i} \iint_\Omega \frac{\partial u}{\partial z} \, \overline{h} \, dz \wedge d\bar{z} \\
&= -\frac{1}{2i} \int_{b\Omega} u \, \overline{h} \, d\bar{z} = -\frac{1}{2i} \int_{b\Omega} (\mathcal{E}u) \, \overline{h} \, d\bar{z} \\
&= -\frac{1}{2i} \iint_\Omega \frac{\partial \mathcal{E} u}{\partial z} \, \overline{h} \, dz \wedge d\bar{z} \\
&= \iint_\Omega G(z) \, \overline{h(z)} \, dx \wedge dy
\end{aligned}$$

where $G = (\partial/\partial z)(\mathcal{E}\Lambda v)$ is in $A^\infty(\Omega)$. We have shown that

$$\langle v, h \rangle_\Omega = \langle G, h \rangle_\Omega$$

for all $h \in A^\infty(\Omega)$. But $A^\infty(\Omega)$ is dense in the Bergman space (Corollary 15.1). Hence, it follows that this inner product identity holds for all h in the Bergman space and it follows that $G = Bv$. The proof of the identity is complete.

If we combine formula (17.1) with Theorem 14.3, we obtain the following result, that we will need in Chapter 22.

Theorem 17.1. *The Bergman projection associated to a bounded domain with smooth real analytic boundary maps compactly supported functions in L^2 to holomorphic functions that extend to be holomorphic on a neighborhood of the closure.*

Indeed, the operator Λ maps compactly supported functions to functions that are holomorphic in a neighborhood of the boundary, then $\mathcal{E}$ maps such functions to functions that are are harmonic on a neighborhood of the closure of the domain, and finally, $\partial/\partial z$ of the end result produces a function holomorphic on a neighborhood of the closure.

In the simply connected case, the holomorphic part of $\mathcal{E}u$ is explicit via Theorem 10.1 and we obtain

$$Bv = \frac{\partial}{\partial z}\left(\frac{P(S_a \Lambda v)}{S_a}\right).$$

In the n-connected case, the formulas developed in Chapter 14 show that linear combinations of the functions $\log|z - b_i|$ enter the picture. The formulas are uglier, but not too difficult to write out. We omit the details.

The Cauchy transform was very important in the study of the Hardy space where it served to expose properties of the Szegő projection. The relationship between the operator Λ and the Bergman projection leads us to define a kind of Cauchy transform that is relevant to the Bergman space and projection. The improved Cauchy integral formula in Theorem 2.1 decomposes a function into two pieces. The first piece, that is an integral over the boundary, gives rise to the Cauchy transform. We now define an operator that comes from the second piece, the integral over the interior of Ω in that formula. If $U \in L^2(\Omega)$, we define the *solid Cauchy transform* $\mathcal{C}_s U$ of U to be the antiholomorphic function on the complement of $\overline{\Omega}$ given by

$$(\mathcal{C}_s U)(z) = \frac{1}{2\pi i}\iint_{\zeta\in\Omega} \frac{U(\zeta)}{\bar{\zeta} - \bar{z}}\, d\zeta \wedge d\bar{\zeta} \qquad \text{for } z \notin \overline{\Omega}.$$

In Chapter 3, we saw that the Cauchy integral had well defined boundary values. A similar result is true for the solid Cauchy transform.

Theorem 17.2. *If Ω is a bounded domain with C^∞ smooth boundary, then the solid Cauchy transform maps $C^\infty(\overline{\Omega})$ into the space of antiholomorphic functions on $\mathbb{C} - \overline{\Omega}$ that vanish at infinity and that extend C^∞ smoothly up to the boundary of Ω.*

Proof. It is clear that the solid Cauchy transform of a bounded function vanishes at infinity. The hard part of the theorem is the smoothness assertion.

Given $U \in C^\infty(\overline{\Omega})$, Lemma 2.1 implies that there is a function $\varphi \in C^\infty(\overline{\Omega})$ that vanishes on $b\Omega$ such that $U - \partial\varphi/\partial z$ vanishes to order m on the boundary. Let $\Psi = U - \partial\varphi/\partial z$. The complex Green's identity reveals that $\mathcal{C}_s(\partial\varphi/\partial z) \equiv 0$, and therefore, it follows that $\mathcal{C}_s U = \mathcal{C}_s \Psi$. Now it is a simple matter to extend Ψ to all of $\mathbb{C}$ by setting it equal to zero outside of Ω, to change variables, and to differentiate under the integral as we did way back in the proof of Theorem 2.2, to see that $\mathcal{C}_s U$ extends to be m times differentiable on $\mathbb{C}$. Since m can be taken to be arbitrarily large, it follows that $\mathcal{C}_s U$ extends C^∞ smoothly to the boundary. $\square$

A byproduct of the proof above is that the boundary values of $\mathcal{C}_s U$ agree with the boundary values of the function ΛU defined above. Consequently, it follows that the Bergman projection can be expressed via $Bv = (\partial/\partial z)(\mathcal{E}\mathcal{C}_s v)$. To better understand this identity, we will examine the solid Cauchy transform in more detail.

The solid Cauchy transform and the classical Cauchy transform are adjoint operators in a sense that we will now explain. We will prove that if $U \in C^\infty(\overline{\Omega})$ and $v \in C^\infty(b\Omega)$, then

$$\langle \mathcal{C}_s U, v \rangle_{b\Omega} = \langle U, 2i\mathcal{C}(v\overline{T}\,)\rangle_\Omega.$$

Indeed, using the same notation that we used in the proof of Theorem 17.2, we may write $\mathcal{C}_s U = \mathcal{C}_s \Psi$ and

$$\langle \mathcal{C}_s U, v \rangle_{b\Omega} = \int_{z\in b\Omega} \frac{1}{2\pi i}\left(\iint_{\zeta\in\Omega} \frac{\Psi(\zeta)}{\bar\zeta - \bar z}\, d\zeta \wedge d\bar\zeta\right) \overline{v(z)}\; ds.$$

Since Ψ can be assumed to vanish to high order on the boundary, we may use Fubini's theorem as we did in the proof of the adjoint identity (3.2) for the Cauchy transform to see that the last integral is equal to

$$\iint_{\zeta\in\Omega} \Psi(\zeta)\left(\frac{1}{2\pi i}\int_{z\in b\Omega} \frac{\overline{v(z)}}{\bar\zeta - \bar z}\, ds\right) d\zeta\wedge d\bar\zeta = \langle \Psi, 2i\mathcal{C}(v\overline{T}\,)\rangle_\Omega.$$

Finally, since $\Psi = U - \partial\varphi/\partial z$ and since, by Lemma 15.1, functions of the form $\partial\varphi/\partial z$ with φ vanishing on the boundary are orthogonal to $H^2(\Omega)$, we may reduce this last inner product to $\langle U, 2i\mathcal{C}(v\overline{T}\,)\rangle_\Omega$, and we have proved that $\langle \mathcal{C}_s U, v\rangle_{b\Omega} = \langle U, 2i\mathcal{C}(v\overline{T}\,)\rangle_\Omega$.

This adjoint property will allow us to extend the solid Cauchy transform to be an operator from $L^2(\Omega)$ to $L^2(b\Omega)$. Recall that Theorem 6.5

implies that there is a constant C such that $\|\mathcal{C}(v\overline{T})\|_\Omega \le C\|v\overline{T}\|_{b\Omega} = C\|v\|_{b\Omega}$. Hence, if $U \in C^\infty(\overline{\Omega})$ and $v \in C^\infty(b\Omega)$, then

$$|\langle \mathcal{C}_s U, v\rangle_{b\Omega}| = |\langle U, 2i\mathcal{C}(v\overline{T}\,)\rangle_\Omega| \le 2C\|U\|_\Omega \|v\|_{b\Omega}.$$

Since $C^\infty(b\Omega)$ is dense in $L^2(b\Omega)$, it follows that

$$\|\mathcal{C}_s U\|_{b\Omega} = \sup\{|\langle \mathcal{C}_s U, v\rangle_{b\Omega}| \,:\, v \in C^\infty(b\Omega) \text{ and } \|v\|_{b\Omega} = 1\},$$

and therefore, we see that $\|\mathcal{C}_s U\|_{b\Omega} \le 2C\|U\|_\Omega$. Now, since $C^\infty(\overline{\Omega})$ is dense in $L^2(\Omega)$, this estimate shows that $\mathcal{C}_s$ extends uniquely as an operator from $L^2(\Omega)$ to $L^2(b\Omega)$. Let us state this as a theorem.

Theorem 17.3. *If Ω is a bounded domain with C^∞ smooth boundary, then the solid Cauchy transform extends to be a bounded operator from $L^2(\Omega)$ to $L^2(b\Omega)$.*

The solid Cauchy transform gives rise to a method for measuring smoothness of holomorphic functions. We know that if $h \in A^\infty(\Omega)$, then $\mathcal{C}_s h$ is an antiholomorphic function on the complement of $\overline{\Omega}$ that extends C^∞ smoothly to the boundary of Ω. We will now show that if $h \in H^2(\Omega)$ is such that $\mathcal{C}_s h$ extends smoothly to $b\Omega$, then it must be that $h \in A^\infty(\Omega)$.

Theorem 17.4. *Suppose that Ω is a bounded domain with C^∞ smooth boundary. A function h in the Bergman space is also in $A^\infty(\Omega)$ if and only if*

$$\sup_{w \notin \overline{\Omega}} \left| \iint_{z\in\Omega} \frac{h(z)}{(\bar{z}-\bar{w})^n}\, dA \right| < \infty$$

for each positive integer n, i.e., if and only if the solid Cauchy transform of h extends C^∞ smoothly to $b\Omega$.

The reader might imagine that the problem of testing the finiteness of the integrals in Theorem 17.4, if not impossible, should be at least as difficult as estimating bounds for the derivatives of h directly. To show that this may not be the case, we will give a simple application of the theorem before proving it. Given a point $a \in \Omega$, let $h(z) = K_a(z)$ where K_a denotes the Bergman kernel function. The reproducing property of the kernel yields that the integral in the statement of the theorem is simply $(\bar{a}-\bar{w})^{-n}$, which is clearly bounded as a function of w on the complement of $\overline{\Omega}$ for each positive integer n. Hence, Theorem 17.4 leads to a two line proof that the Bergman kernel $K(z,w)$ is in $A^\infty(\Omega)$ as a function of z for each fixed $w \in \Omega$. Theorem 17.4 can also be used to give short proofs that conformal and proper holomorphic maps between smooth domains must extend smoothly to the boundary.

We now turn to the proof of Theorem 17.4.