Proof. The idea of the proof is simple. If $h \in A^\infty(\Omega)$, we know that

$$h = Bh = \frac{\partial}{\partial z}\mathcal{E}C_s h,$$

and this formula should extend to remain valid if h is merely assumed to be in the Bergman space. Hence, it should follow that if $C_s h$ has smooth boundary values, then h must be in $A^\infty(\Omega)$ because the other operators in the formula for the Bergman projection behave well when acting on C^∞ smooth functions. The idea is simple, and it is going to work. However, some of the points in the argument that seem obvious require proof.

Suppose that h is a function in the Bergman space such that all the suprema in the hypothesis of the theorem are finite. This means that $\mathcal{C}_s h$ defines an antiholomorphic function on the complement of $\overline{\Omega}$ that extends C^∞ smoothly to $b\Omega$. The first obvious sounding step in the proof requiring justification concerns the boundary values of $\mathcal{C}_s h$. When h is merely in the Bergman space, the definition of $\mathcal{C}_s h$ as an element of $L^2(b\Omega)$ was made by means of a sequence of functions $h_i \in A^\infty(\Omega)$ such that $h_i \to h$ in $L^2(\Omega)$. Then $\mathcal{C}_s h_i \to \mathcal{C}_s h$ in $L^2(b\Omega)$. We must show that $\mathcal{C}_s h$ as defined as a limit of a sequence agrees with the actual C^∞ boundary values of $\mathcal{C}_s h$, treated as a function on the complement of $\overline{\Omega}$. Fortunately, the groundwork has been laid in Chapter 6. Let $F(z) = 1/z$ and consider the set $\widetilde{\Omega}$ given as the image of the complement of $\overline{\Omega}$ under the map F. We also assume that the origin is in Ω so that the set $\widetilde{\Omega}$ is a bounded domain with C^∞ smooth boundary. By composing with F^{-1}, we may consider $\mathcal{C}_s h_i$ and $\mathcal{C}_s h$ to be antiholomorphic functions on $\widetilde{\Omega}$ such that the boundary values of $\mathcal{C}_s h_i$ converge in $L^2(b\widetilde{\Omega})$ to a function H in that space. Note that we are using the fact that $\mathcal{C}_s$ maps $L^2(\Omega)$ to functions vanishing at ∞ and hence $0 \in \widetilde{\Omega}$, is a removable singularity for the antiholomorphic functions on $\widetilde{\Omega}$ under study. Now, the functions $\mathcal{C}_s h_i$ converge uniformly on compact subsets to $\mathcal{C}_s h$ and it follows from Theorem 6.3 that H is equal to the smooth boundary values of $\mathcal{C}_s h$. Thus, the first sticky point in the proof is past; we have shown that $\mathcal{C}_s h$, as an element of $L^2(b\Omega)$, is in $C^\infty(b\Omega)$.

Next, we must show that the operator $(\partial/\partial z)\mathcal{E}$ can be defined to make sense as an operator on $L^2(b\Omega)$. Let a be a point in Ω, and suppose that u is a harmonic function on Ω that is in $C^\infty(\overline{\Omega})$. Now $\partial u/\partial z$ is in $A^\infty(\Omega)$ and

$$\begin{aligned}\frac{\partial u}{\partial z}(a) &= \iint_\Omega \frac{\partial u}{\partial z}\,\overline{K_a}\; dA \\ &= -\frac{1}{2i}\iint_\Omega \frac{\partial u}{\partial z}\,\overline{K_a}\; dz \wedge d\bar{z} = -\frac{1}{2i}\int_{b\Omega} u\,\overline{K_a}\; d\bar{z},\end{aligned}$$

where we have used the reproducing property of the Bergman kernel K_a and the complex Green's identity. Hence, it follows that

$$\left|\frac{\partial u}{\partial z}(a)\right| \leq \frac{1}{2}\|u\|\|K_a\|$$

where the norms denote $L^2(b\Omega)$ norms. We know that K_a is in $A^\infty(\Omega)$; hence $\|K_a\|$ is well defined and finite. This shows that the formula $h(a) = (\partial/\partial z)(\mathcal{E}\mathcal{C}_s h)(a)$ makes sense and is stable under the limiting process of taking h_i tending to h in $L^2(\Omega)$. Finally, we may state, therefore, that h is in $A^\infty(\Omega)$. □

Now that we have studied the boundary behavior of the Cauchy transform and the solid Cauchy transform, we can look at the improved Cauchy integral formula in Theorem 2.1 in a new light. Given a function u in $C^\infty(b\Omega)$, we can find a function $U \in C^\infty(\overline{\Omega})$ such that $U = u$ on the boundary. If we write out the improved Cauchy integral formula for U and restrict to the boundary, we obtain the following classical theorem.

Theorem 17.5. *If Ω is a bounded domain with C^∞ smooth boundary, then every function u in $C^\infty(b\Omega)$ can be decomposed uniquely as a sum $h + H$ where $h \in A^\infty(\Omega)$ and H is a holomorphic function on the complement of $\overline{\Omega}$ that vanishes at infinity and that extends C^∞ smoothly to $b\Omega$.*

We have proved that there is a decomposition $u = h + H$. To see that the decomposition is unique, suppose that $h_1 + H_1$ and $h_2 + H_2$ are two such decompositions. Then $h_1 - h_2 = H_2 - H_1$ on the boundary, and Morera's Theorem can be used to see that the continuous function on the whole complex plane that is defined to be equal to $h_1 - h_2$ in Ω and equal to $H_2 - H_1$ outside Ω is an entire function. Since this entire function vanishes at infinity, it must be the zero function, and uniqueness is proved.

18

The classical Neumann problem

The Dirichlet problem is very important in the study of harmonic functions and conformal mapping. Equally important is the classical Neumann problem. Suppose Ω is a bounded domain with C^∞ smooth boundary. Given a function ψ in $C^\infty(b\Omega)$, the Neumann problem is to find a function $\varphi \in C^\infty(\overline{\Omega})$ that is harmonic on Ω such that the normal derivative of φ on the boundary is equal to ψ. We will use the notation $\partial\varphi/\partial n$ to denote the normal derivative of φ. The first observation to be made is that not every ψ can be equal to the normal derivative of a harmonic function. Indeed, if φ is harmonic, then Gauss' theorem yields

$$\int_{b\Omega} (\partial\varphi/\partial n)\ ds = \iint_{\Omega} \Delta\varphi\ dA = 0.$$

Thus, it is necessary to stipulate that $\int_{b\Omega} \psi\, ds = 0$.

We will prove that the Neumann problem can be solved by expressing the solution in terms of the Szegő projections of explicit functions. In the case that Ω is simply connected, the result can be stated as follows.

Theorem 18.1. *Suppose Ω is a bounded simply connected domain with C^∞ smooth boundary, and let $a \in \Omega$ be given. Suppose ψ is a function in $C^\infty(b\Omega)$ satisfying $\int_{b\Omega} \psi\, ds = 0$. Then, a solution to the Neumann problem*

$$\begin{aligned} \Delta\varphi = 0 \qquad & \textit{on } \Omega \\ \frac{\partial\varphi}{\partial n} = \psi \qquad & \textit{on } b\Omega \end{aligned}$$

is given by $\varphi = h + \overline{H}$ where h and H are holomorphic functions on Ω defined via

$$\begin{aligned} h' &= S_a\, P\left(\psi / \overline{L_a}\right), \ \textit{and} \\ H' &= L_a\, P\left(\overline{\psi} / \overline{S_a}\right). \end{aligned}$$

Furthermore, the solution φ is in $C^\infty(\overline{\Omega})$.

We remark that, in the setting of Theorem 18.1, S_a is nonvanishing on

$\overline{\Omega}$. It is also worth pointing out that, even though L_a has a simple pole at a, the expression in the statement of the theorem for H' is holomorphic near a because, using the fact that $S(a,z)=\overline{S(z,a)}$, we have

$$P\left(\overline{\psi}/\overline{S_a}\right)(a)=\int_{z\in b\Omega}S(a,z)\frac{\overline{\psi}}{\overline{S(z,a)}}\,ds=\int_{b\Omega}\overline{\psi}\,ds=0.$$

We will state a version of this theorem for multiply connected domains in Chapter 20. It is somewhat more involved because not every harmonic function on a multiply connected domain can be expressed as $h+\overline{H}$ and not every holomorphic function is equal to the derivative of a holomorphic function.

It is not hard to show that the solution to the Neumann problem is unique up to the addition of an arbitrary constant. Indeed, if φ is a harmonic function in $C^\infty(\overline{\Omega})$ such that $\partial\varphi/\partial n\equiv 0$, then Green's identity implies that

$$0=\int_{b\Omega}\varphi\frac{\partial\varphi}{\partial n}\,ds=\iint_{\Omega}|\nabla\varphi|^2\,dA$$

and this shows that φ must be constant on Ω. Consequently, the derivatives h' and H' in the decomposition of the solution in Theorem 18.1 are uniquely determined, and therefore, the expressions for h' and H' do not depend on the choice of a.

It can be shown that the functions defined in the statement of Theorem 18.1 make sense even when ψ is merely assumed to be continuous on $b\Omega$ and that the normal derivative of $\varphi=h+\overline{H}$ exists in a weak sense and is equal to ψ. We will not study the more subtle problems that arise in the study of harmonic functions with continuous boundary normal derivatives in this book.

Proof of Theorem 18.1. To motivate the proof, let us suppose for the moment that we know that a solution φ exists that is in $C^\infty(\overline{\Omega})$. Any harmonic function on a simply connected domain can be expressed as $h+\overline{H}$ where h and H are holomorphic functions that are determined up to additive constants. Furthermore, since $\varphi\in C^\infty(\overline{\Omega})$, it follows from differentiating $\varphi=h+\overline{H}$ with respect to z and $\bar{z}$ that h and H must be in $A^\infty(\Omega)$. If $z(s)$ parameterizes the boundary of Ω in the standard sense with respect to arc length s, then $(d/ds)h(z(s))=h'(z(s))z'(s)=h'(z(s))T(z(s))$. Since h satisfies the Cauchy-Riemann equations on the boundary, the normal derivative $\partial h/\partial n$ is equal to $-i$ times the derivative of h in the tangential direction pointing in the standard sense. Thus, $\partial h/\partial n=-iT(z)h'(z)$. Similarly, $\partial\overline{H}/\partial n=i\,\overline{T(z)H'(z)}$. Hence,

$$\psi=\frac{\partial\varphi}{\partial n}=-iT(z)h'(z)+i\,\overline{T(z)H'(z)}. \tag{18.1}$$

At this point, the orthogonal decomposition of Theorem 4.3 should spring to mind. We may use formula (7.1) to express T as

$$T = i\frac{\overline{L_a}}{S_a},$$

and we may plug this expression into (18.1) and divide by $\overline{L_a}$ to obtain

$$\frac{\psi}{\overline{L_a}} = \frac{h'}{S_a} + i\,\overline{T}\,\frac{\overline{H'}}{\overline{L_a}}. \tag{18.2}$$

Now, because of the nonvanishing of S_a and L_a on $\overline{\Omega}$, it follows that h'/S_a and H'/L_a are both in $A^\infty(\Omega)$ (the pole of L_a implies that the second function vanishes at a, but this is of no consequence to us). Hence, (18.2) is an orthogonal sum, and we obtain

$$\frac{h'}{S_a} = P\left(\psi/\overline{L_a}\right).$$

Furthermore,

$$-i\,\frac{H'}{L_a} = P\left(\overline{T}\frac{\overline{\psi}}{L_a}\right),$$

and, using the fact that $\overline{T}/L_a = 1/(i\,\overline{S_a}\,)$ by (7.1), we see that

$$\frac{H'}{L_a} = P\left(\overline{\psi}/\,\overline{S_a}\,\right).$$

To prove Theorem 18.1, we need only trace this argument backwards. Indeed, define $\varphi = h + \overline{H}$ where h and H are holomorphic functions on Ω whose derivatives satisfy

$$h' = S_a\,P\left(\psi/\,\overline{L_a}\,\right), \text{ and}$$
$$H' = L_a\,P\left(\overline{\psi}/\,\overline{S_a}\,\right).$$

Then φ is in $C^\infty(\overline{\Omega})$ and

$$\frac{\partial\varphi}{\partial n} = -iT(z)h'(z) + i\,\overline{T(z)H'(z)} = -iS_aP(\psi/\,\overline{L_a}\,)T + i\,\overline{L_aP(\,\overline{\psi}/\,\overline{S_a}\,)T}.$$

The proof will be finished after we prove the following lemma which holds even when Ω is not assumed to be simply connected. □

Lemma 18.1. *If Ω is a bounded finitely connected domain with C^∞ smooth boundary and if $\psi \in L^2(b\Omega)$, then*

$$\psi = -iS_aP(\psi/\,\overline{L_a}\,)T + i\,\overline{L_aP(\,\overline{\psi}/\,\overline{S_a}\,)T}.$$

Proof of the lemma. Divide the equation in the statement of the lemma by $\overline{L_a}$, and use identity (7.1) to see that our problem is equivalent to proving that

$$\frac{\psi}{\overline{L_a}} = P(\psi/\,\overline{L_a}\,) + i\,\overline{P(\,\overline{\psi}/\,\overline{S_a}\,)T}.$$

This is an orthogonal sum. The holomorphic part is obviously what it should be. The proof of the lemma will be finished if we show that

$$P^{\perp}\left(\frac{\psi}{\overline{L_a}}\right) = i\,\overline{P(\,\overline{\psi}/\,\overline{S_a}\,)T}.$$

Since $P^{\perp}u = \overline{P(\overline{uT})T}$, this is equivalent to showing that

$$P\left(\frac{\overline{\psi}}{L_a}\overline{T}\right) = -iP(\,\overline{\psi}/\,\overline{S_a}\,),$$

and this follows directly from identity (7.1). The proof of the lemma is done and therefore, so is the proof of the theorem. □

19

Harmonic measure and the Szegő kernel

The expression $-ih'T + i\,\overline{H'T}$ appears as the normal derivative of $h + \overline{H}$ in the proof of Theorem 18.1. In a simply connected domain, we will see that the condition $-ih'T + i\,\overline{H'T} = 0$ forces $h' \equiv 0$ and $H' \equiv 0$. Before we treat the Neumann problem in a multiply connected domain, we must determine which functions h and H in $H^2(b\Omega)$ satisfy the condition $-ihT + i\,\overline{HT} = 0$, i.e., satisfy $hT = \overline{HT}$. By Theorem 4.3, a function u that is expressible as both hT and $\overline{HT}$ would be orthogonal to $H^2(b\Omega)$ and orthogonal to conjugates of functions in $H^2(b\Omega)$. We will see that such functions are closely related to the classical harmonic measure functions (which we define below).

In this chapter, we will be dealing with a bounded n-connected domain Ω with C^∞ smooth boundary curves. Let $\{\gamma_j\}_{j=1}^n$ denote the n boundary curves of Ω. For convenience, we may assume that γ_n is the outer boundary curve, i.e., the one that also bounds the unbounded component of the complement of Ω in $\mathbb{C}$. The harmonic measure functions $\{\omega_j\}_{j=1}^n$ associated to Ω are defined as follows. The function ω_j is equal to the harmonic function on Ω that solves the Dirichlet problem with boundary data equal to one on γ_j and equal to zero on the other boundary curves. Note that we know that these harmonic measure functions are in $C^\infty(\overline{\Omega})$ by Theorem 14.2.

An important holomorphic function associated to ω_j is the function $F_j' = 2\partial\omega_j/\partial z$. The prime in the notation F_j' is traditional; F_j' is the derivative of a multi-valued holomorphic function. To see this, suppose that v is a harmonic conjugate for ω_j on a small disc contained in Ω and define $F_j = \omega_j + iv$ there. By the Cauchy-Riemann equations, $F_j' = (\partial/\partial x)\omega_j - i(\partial/\partial y)\omega_j = 2\partial\omega_j/\partial z$. Thus, F_j' is the derivative of the multi-valued function obtained by analytically continuing around Ω a germ of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j.

Our next theorem will show how the functions F_j' are related to the Szegő kernel and its zeroes. Let $a \in \Omega$ be a given point. By Theorem 13.1, the function $S_a(z) = S(z, a)$ has exactly $n-1$ zeroes in Ω (counted with multiplicity). We will prove in Chapter 27 that the zeroes of $S(z, a)$ become simple zeroes as a approaches the boundary of Ω. We assume now that a has been chosen so that the zeroes of $S(z,a)$ are all *simple* zeroes. Let $\{a_j\}_{j=1}^{n-1}$ denote these distinct $n-1$ zeroes of S_a.

Let $\mathcal{Q}$ denote the space of functions in $L^2(b\Omega)$ that are orthogonal to both the Hardy space $H^2(b\Omega)$ and to the space of functions that are complex conjugates of functions in $H^2(b\Omega)$.

Let $\mathcal{F}'$ denote the (complex) linear span of $\{F_j' : j = 1, 2, \ldots, n-1\}$. Although the function F_n' is not included in the spanning set for $\mathcal{F}'$, this function is in $\mathcal{F}'$. Indeed, the maximum principle shows that $\sum_{j=1}^n \omega_j \equiv 1$ and consequently, $\sum_{j=1}^n F_j' \equiv 0$.

Theorem 19.1. *The space $\mathcal{Q}$ of functions in $L^2(b\Omega)$ orthogonal to both $H^2(b\Omega)$ and the space of functions that are conjugates of $H^2(b\Omega)$ functions is equal to $\{hT : h \in \mathcal{F}'\}$. Furthermore, $\mathcal{F}'$ is equal to the complex linear span of*

$$\{L(z,a_j)S(z,a) : j = 1, 2, \ldots, n-1\}.$$

$\mathcal{F}'$ is also equal to the complex linear span of

$$\{L(z,a)S(z,a_j) : j = 1, 2, \ldots, n-1\}.$$

The part of this theorem that states that $\mathcal{F}'$ is equal to the span of $L(z,a_j)S(z,a)$ was proved by Schiffer in [Sch].

Proof. Notice that the functions $L(z,a_j)S(z,a)$ are in $A^\infty(\Omega)$ because the pole of $L(z,a_j)$ at a_j is cancelled by the zero of $S(z,a)$ at a_j. Similarly, the functions $L(z,a)S(z,a_j)$ are in $A^\infty(\Omega)$ because the pole of $L(z,a)$ at a is cancelled by the zero of $S(z,a_j)$ at a. Let $\mathcal{L}$ denote the complex linear span of $\{L(z,a_j)S(z,a) : j = 1, 2, \ldots, n-1\}$. We make the convention that, if $\mathcal{G}$ is a subset of $H^2(b\Omega)$, then $\mathcal{G}T$ will denote the set of functions of the form gT, where $g \in \mathcal{G}$. We will prove the theorem by first showing that $\mathcal{F}'T \subset \mathcal{Q} \subset \mathcal{L}T$. The dimension of $\mathcal{L}T$ as a vector space over the complex numbers is less than or equal to $(n-1)$. If we show that $\mathcal{F}'T$ is a vector space of dimension $(n-1)$, it follows that $\mathcal{F}'T = \mathcal{Q} = \mathcal{L}T$.

That $\mathcal{F}'T \subset \mathcal{Q}$ follows from Theorem 4.3 and the identity,

$$F_j'T = -\overline{F_j'T} \qquad \text{on } b\Omega, \tag{19.1}$$

which we now prove. This is an important fact. We will also need it in the next chapter when we study the Neumann problem in multiply connected domains. Let $z(t)$ parameterize a boundary curve of Ω. Since ω_j is constant on boundary curves, we have

$$0 = \frac{d}{dt}\omega_j(z(t)) = \frac{\partial \omega_j}{\partial z}\, z'(t) + \frac{\partial \omega_j}{\partial \bar{z}}\, \overline{z'(t)}.$$

Since $(\partial\omega_j/\partial z)$ is equal to the complex conjugate of $(\partial\omega_j/\partial\bar{z})$, identity (19.1) is proved.

We will next prove that $\mathcal{Q} \subset \mathcal{L}T$. If $u \in \mathcal{Q}$, then by Theorem 4.3, u can be written

$$u = hT = \overline{HT},$$

where h and H are elements of $H^2(b\Omega)$. Using identity (7.1) in the form $T = i\,\overline{L_a}/S_a$, we obtain

$$ih\frac{\overline{L_a}}{S_a} = \overline{HT},$$

and therefore,

$$i\frac{h}{S_a} = \frac{\overline{H}}{\overline{L_a}}\overline{T}.$$

Now, H/L_a is a holomorphic function in $H^2(b\Omega)$. Hence, the function on the right hand side of the last equation is orthogonal to $H^2(b\Omega)$. Therefore,

$$iP^{\perp}\left(\frac{h}{S_a}\right) = \frac{\overline{H}}{\overline{L_a}}\overline{T}.$$

But

$$i\frac{h(z)}{S_a(z)} = G(z) + \sum_{j=1}^{n-1} c_j \frac{1}{z - a_j},$$

where $G(z)$ is in $H^2(b\Omega)$ and $c_j = ih(a_j)/S'(a_j, a)$, the prime denoting differentiation in the holomorphic variable. Therefore, since $P^{\perp}G = 0$, we have

$$\frac{\overline{H}}{\overline{L_a}}\overline{T} = \sum_{j=1}^{n-1} c_j P^{\perp}\left(\frac{1}{z - a_j}\right).$$

It is a consequence of the orthogonal decomposition of the Cauchy kernel that we used to define the Garabedian kernel that

$$P^{\perp}\left(\frac{1}{z - a_j}\right) = 2\pi L(z, a_j).$$

Hence,

$$\frac{\overline{H}}{\overline{L_a}}\overline{T} = \sum_{j=1}^{n-1} c_j 2\pi L(z, a_j).$$

Finally, we may multiply through by $\overline{L_a}$ and use identity (7.1) in the form $\overline{L_a} = -iS_aT$ to obtain

$$hT = \overline{HT} = \sum_{j=1}^{n-1} -ic_j 2\pi L(z, a_j) S(z, a) T(z).$$

Hence $h = \sum_{j=1}^{n-1} -ic_j 2\pi L(z,a_j)S(z,a)$ and we have proved that $\mathcal{Q} \subset \mathcal{L}T$.

Next, we show that the dimension of $\mathcal{F}'$ is $(n-1)$ by proving that the functions F_j', $j = 1,\ldots,n-1$ are linearly independent. To see this, we will prove the classical fact that the $(n-1)\times(n-1)$ matrix of periods $[A_{jk}]$ defined via

$$A_{jk} = \int_{\gamma_j} F_k'\,dz, \qquad j = 1,\ldots,n-1,$$

is nonsingular. It is proved in most books on elementary complex analysis that a function $g \in A^\infty(\Omega)$ is the derivative of a holomorphic function on the multiply connected domain Ω if and only if the periods

$$\int_{\gamma_j} g\,dz$$

vanish for $j = 1,2,\ldots n-1$ (see [Ah, p. 146]). Suppose that there exist constants c_k, not all zero, so that $\sum_{k=1}^{n-1} A_{jk}c_k = 0$ for each j. Then it follows that all the periods of the function $\sum_{k=1}^{n-1} c_k F_k'$ vanish, and so $G' = \sum_{k=1}^{n-1} c_k F_k'$ for some $G \in A^\infty(\Omega)$. Now, $G' = 2(\partial\omega/\partial z)$ where $\omega = \sum_{k=1}^{n-1} c_k\omega_k$. Since not all the c_k are zero, and since $\omega = 0$ on γ_n, it is clear that the function ω cannot be an anti-holomorphic function. Hence, G' is not identically zero and therefore $\iint_\Omega |G'|^2\,dA \neq 0$. But

$$\frac{1}{4}\iint_\Omega |G'|^2\,dA = \iint_\Omega \frac{\partial\omega}{\partial z}\frac{\partial\bar{\omega}}{\partial\bar{z}}\,dA = \frac{1}{2i}\int_{b\Omega} \bar{\omega}\frac{\partial\omega}{\partial z}\,dz$$

by the complex Green's identity. If we expand this last integral, we obtain

$$\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} \bar{c}_j\,c_k \int_{b\Omega} \overline{\omega_j}\frac{\partial\omega_k}{\partial z}\,dz$$

and this last sum is a linear combination of the numbers $\sum_{k=1}^{n-1} A_{jk}c_k$, which we have assumed to be zero. This contradiction shows that the matrix of periods is nonsingular, and therefore, the proof that $\mathcal{F}'T = \mathcal{Q} = \mathcal{L}T$ is complete.

To finish the proof, we must see that $\mathcal{L}$ is equal to the complex linear span $\widetilde{\mathcal{L}}$ of $\{L(z,a)S(z,a_j) : j = 1,2,\ldots,n-1\}$. The space $\mathcal{Q}$ of functions in $L^2(b\Omega)$ orthogonal to $H^2(b\Omega)$ and conjugates of functions in $H^2(b\Omega)$ is, by its definition, invariant under complex conjugation. By (7.1), we may write

$$L(z,a_j)S(z,a)T(z) = -\overline{S(z,a_j)L(z,a)T(z)}. \tag{19.2}$$

Hence, $\mathcal{Q}$ is equal to the space of conjugates of functions in $\mathcal{L}T$, which by (19.2) is equal to $\widetilde{\mathcal{L}}T$. Since, $\mathcal{Q} = \mathcal{L}T$, it follows that $\mathcal{L} = \widetilde{\mathcal{L}}$, and the proof is complete. □

When we study the Neumann problem in multiply connected domains, we will need to know the following classical fact about the functions F'_j. It is a direct consequence of the result proved above that the matrix of periods $[A_{jk}]$ is nonsingular.

Lemma 19.1. *Given a holomorphic function $h \in A^\infty(\Omega)$, there exist constants c_j and another holomorphic function $H \in A^\infty(\Omega)$ such that $h = H' + \sum_{k=1}^{n-1} c_k F'_k$.*

Thus every function in $A^\infty(\Omega)$ is the derivative of a function in $A^\infty(\Omega)$ modulo the space of linear combinations of the F'_j.

Since $[A_{jk}]$ is nonsingular, there exist complex constants c_j, $j = 1, \ldots, n-1$ satisfying

$$\int_{\gamma_j} h\, dz = \sum_{k=1}^{n-1} A_{jk} c_k, \qquad j = 1, \ldots, n-1.$$

Now, all the periods of $h - \sum_{k=1}^{n-1} c_k F'_k$ vanish, and so this function is equal to the derivative of a holomorphic function H on Ω. It is clear that $H \in A^\infty(\Omega)$ and the lemma is proved.

We close this chapter by studying a classical application of the harmonic measure functions to the conformal classification of multiply connected domains. Given an n-connected domain Ω with C^∞ smooth boundary, we describe the construction of a biholomorphic map of Ω onto a canonical n-connected domain that is an annulus minus $n-2$ disjoint closed circular arcs. The annulus and each of the closed circular arcs are centered at the origin and the $n-2$ disjoint arcs each span angles that are less than 2π. Lemma 12.1 allows us to reduce the problem of constructing such a map to the case in which Ω has real analytic boundary. One of the virtues of such a domain is that its harmonic measure functions ω_k extend to be harmonic on a neighborhood of $\overline{\Omega}$. Indeed, Theorem 14.2 shows that these functions extend continuously to the boundary, and the Schwarz reflection principle applies to yield the harmonic extensions. Harmonic extendibility of the functions ω_k, in turn, implies that the functions F'_k extend holomorphically to a neighborhood of $\overline{\Omega}$.

The function $f(z)$ we will construct mapping Ω onto an annular domain of the type described above will extend holomorphically past the boundary of Ω, and it will map the outer boundary γ_n of Ω onto the unit circle. To motivate the construction, let us take a moment to see why such an f should be related to the harmonic measure functions. Consider the function $\ln|f(z)|$. It is harmonic on a neighborhood of $\overline{\Omega}$ and constant on each boundary curve of Ω. Hence, there exist real constants $r_1, r_2, \ldots, r_n$ such that

$$\ln|f(z)| = \sum_{k=1}^{n} r_k \omega_k.$$

Since $|f| = 1$ on γ_n, it follows that $r_n = 0$. We now differentiate this equation with respect to z. Notice that $\ln|f(z)| = \frac{1}{2}\ln|f(z)|^2 = \frac{1}{2}\ln f(z)\overline{f(z)}$ and so

$$\frac{\partial}{\partial z}\ln|f(z)| = \frac{1}{2}\frac{1}{f(z)\overline{f(z)}}\frac{\partial}{\partial z}\left(f(z)\overline{f(z)}\right) = \frac{f'(z)}{2f(z)}.$$

Also recall that $(\partial/\partial z)\omega_k = \frac{1}{2}F_k'$. Hence, we obtain

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n-1} r_k F_k'. \tag{19.3}$$

Let us now restrict our attention to a simply connected subdomain Ω_0 of Ω. Let $G(z)$ be a holomorphic antiderivative of $\sum_{k=1}^{n-1} r_k F_k'$ on Ω_0. Notice that f satisfies the ODE, $f' - G'f = 0$, and hence, $(fe^{-G})' \equiv 0$. It follows that $f = ce^G$ on Ω_0 for some choice of a constant c. The key to the construction of f will be to use (19.3) to determine an appropriate choice of the constants r_k, and then to find a multi-valued antiderivative G of $\sum_{k=1}^{n-1} r_k F_k'$ that will make $f = e^G$ a single valued holomorphic function with the desired properties.

To put (19.3) in a more useful form, we integrate it around the boundary curve γ_j. The left hand side of the equation represents $\frac{1}{i}$ times the increase in the argument of $f(z)$ as z traces out γ_j in the standard sense. Let $\Delta_j \arg f$ represent this increase and, as before, let A_{jk} denote the period, $\int_{\gamma_j} F_k'\,dz$. We may rewrite the integrated equation in the form

$$\frac{1}{i}\Delta_j \arg f = \sum_{k=1}^{n-1} A_{jk} r_k, \qquad j = 1, \ldots, n, \tag{19.4}$$

which is an $n \times (n-1)$ linear system. Actually, the n-th equation in the system must be dependent upon the first $(n-1)$ equations because $\sum_{j=1}^{n} A_{jk} = \int_{b\Omega} F_k'\,dz = 0$ by Cauchy's theorem. Keep in mind that we have shown that the submatrix $[A_{jk}]_{j,k=1,\ldots,n-1}$ is nonsingular.

We now describe how to construct from scratch a mapping f like the one we assumed to exist above. The key to the construction is the system

(19.4). We seek a function f that is holomorphic on a neighborhood of $\overline{\Omega}$ mapping the outer boundary of Ω onto the unit circle, mapping γ_1 onto a circle of radius $R_1 < 1$ centered at the origin, and mapping $\gamma_2, \ldots, \gamma_{n-1}$ to circular arcs of radii between R_1 and 1. As z traces out γ_n in the standard sense, we want $f(z)$ to trace out the unit circle in the standard sense, and so we need $\Delta_n \arg f = 2\pi$. Similarly, as z traces out γ_1 in the standard sense (clockwise), we want $f(z)$ to trace out a circle of radius $R_1 < 1$. For f to preserve the sense of angles, it must be that $f(z)$ traces out this circle of radius R_1 in the clockwise sense, and so we need $\Delta_1 \arg f = -2\pi$. As z traces out any of the other $n-1$ boundary curves, we expect $f(z)$ to move back and forth along a circular arc spanning an angle of less than 2π, and so we need $\Delta_j \arg f = 0$ for $j = 2, \ldots, n-1$. Now we may determine constants $\{r_k\}_{k=1}^{n-1}$ using the first $(n-1)$ of the equations in system (19.4) using the increments of arg f around the boundary curves that we just specified. Fortunately, because $2\pi + 0 + \cdots + 0 - 2\pi = 0$, the n-th equation in that system is satisfied because it represents the sum of the first $(n-1)$ equations. Let $r_n = 0$. Identity (19.1) shows that $A_{jk} = -\overline{A_{jk}}$, and so the matrix of periods is a matrix of pure imaginary numbers. Since the inhomogeneous term in the system (19.4) is also pure imaginary, it follows that the constants r_k are *real* numbers.

Pick a point $z_0 \in \Omega$ and, given $z \in \Omega$, let $\Gamma(z_0, z)$ denote a curve in Ω starting at z_0 and ending at $z \in \Omega$. Let $H(z) = \sum_{k=1}^{n} r_k F_k'(z)$ and define $G(z) = \int_{\Gamma(z_0,z)} H(\zeta)\, d\zeta$. Since the periods of H are all integer multiples of $2\pi i$, it follows that, although G may not be uniquely defined on Ω, the real part of G and e^G are. Furthermore, by restricting attention to simply connected subdomains of Ω containing z_0, where we may think of G as being a well defined holomorphic function, it is easily seen that e^G defines a a single valued holomorphic function on Ω. In fact, e^G extends to be holomorphic on a neighborhood of $\overline{\Omega}$.

The mapping we desire is now given as $f = e^G$. To see that the modulus of f is constant on each boundary curve of Ω, let $\omega = \sum_{k=1}^{n-1} r_k \omega_k$. Restrict attention to a simply connected subdomain Ω_0 of Ω containing z_0. We may think of G as being a well defined single valued holomorphic function on Ω_0. We will see that Re $G - \omega$ is a constant on this domain. Since $(\partial/\partial z)\text{Re } h = \frac{1}{2}h'$ for any holomorphic function h, we see that $(\partial/\partial z)(\text{Re } G - \omega) \equiv 0$ on Ω_0, and so Re $G - \omega$ is a real valued antiholomorphic function there. Consequently, Re $G - \omega$ is equal to a real constant λ on Ω_0. Since Re G is well defined on Ω, this constant does not depend on the choice of the subdomain Ω_0. Hence $|e^G| = e^\lambda e^\omega$ on Ω. Let us now redefine f to be equal to ce^G where $c = e^{-\lambda}$. We may now state that $|f| = e^{r_k}$ on γ_k for each $k = 1, \ldots, n$.

The next part of the proof follows directly from the classical argument principle. If w does not lie on any of the circles of radius e^{r_k}, $k = 1, \ldots, n$, then the number of solutions z of the equation $f(z) = w$ is given by $\mathcal{N}(w)$ where

$$\mathcal{N}(w) = \frac{1}{2\pi i}\int_{b\Omega} \frac{f'(\zeta)}{f(\zeta) - w}\,d\zeta = \sum_{k=1}^{n} \mathcal{N}_k(w),$$

where

$$\mathcal{N}_k(w) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{f'(\zeta)}{f(\zeta) - w}\,d\zeta.$$

Our construction guarantees that $\mathcal{N}(0)$ is given by the sum $-1 + 0 + \cdots + 0 + 1$, and so $\mathcal{N}(0) = 0$. Also note that $\mathcal{N}_1(w)$ is equal to -1 if $|w| < e^{r_1}$ and equal to zero if $|w| > e^{r_1}$. Furthermore, $\mathcal{N}_n(w)$ is equal to 1 if $|w| < 1$ and equal to zero if $|w| > 1$. If $k = 2, \ldots, n-1$, then $\mathcal{N}_k(w) = 0$ if $|w| \neq e^{r_k}$. Suppose w is in the image of Ω under f. Because f is an open mapping, we may find a point w_0 in $f(\Omega)$ that is arbitrarily close to w and that does not lie on any of the circles of radii e^{r_k}, $k = 1, \ldots, n$. In order for the numbers $\mathcal{N}_k(w_0)$ to add up to something greater than or equal to one, it must be that $e^{r_1} < |w_0| < 1$. This shows that $f(\Omega)$ is contained in the annulus $A = \{w : e^{r_1} < |w| < 1\}$. Because f is continuous up to the boundary of Ω, it follows that $e^{r_1} \leq e^{r_k} \leq 1$ for $k = 2, \ldots, n-1$. Because the argument of f has been constructed to have a net change of 2π on γ_n and γ_1, we know that f maps γ_1 *onto* the circle of radius e^{r_1} and f maps γ_n *onto* the unit circle. We also know that there is exactly one point z_0 in Ω satisfying $f(z_0) = w_0$ when w_0 is a point in the annulus A that does not lie on any of the circles $|w| = e^{r_k}$.

To finish the proof, we must use the generalized version of the argument principle that we described in Chapter 13. If h is a holomorphic function defined on γ_k, let $\{z_1, z_2, \ldots, z_M\}$ denote the finite set of zeros of h on γ_k. Let $\gamma_k(\epsilon)$ denote the set of curves formed by removing the segments $\gamma_k \cap D_\epsilon(z_i)$, $i = 1, \ldots, M$, each segment parameterized in the same sense as γ_k. We define $\Delta_k \arg h$ to be the limit as ϵ tends to zero of the sum of the increases in arg h over the segments in $\gamma_k(\epsilon)$. The proof that this limit is well defined is contained in the proof given in Chapter 13 of the generalized argument principle.

Consider the triangle with vertices at the origin, the point R, and a point $Re^{i\theta}$, $0 \leq \theta < 2\pi$. It is a simple exercise in high school geometry to see that θ is equal to $-\pi$ plus two times the angle from the real axis to the line segment joining R to $Re^{i\theta}$. Suppose $|w_0| = e^{r_k}$. The geometric fact just mentioned shows that

$$d\arg\,(f(z) - w_0) = \frac{1}{2} d\arg\, f(z) \qquad \text{on } \gamma_k$$

near points z where $f(z) \neq w_0$, and this shows that

$$\frac{1}{2\pi}\Delta_k \arg\,(f(z) - w_0) = \frac{1}{2}\mathcal{N}_k(0).$$

If $|w_0| \neq e^{r_j}$, then $f(z) - w_0$ has no zeroes on γ_j and so $\frac{1}{2\pi}\Delta_j \arg\,(f(z) - w_0) = \mathcal{N}_j(w_0)$. The generalized argument principle states that the number of solutions to $f(z) = w_0$ with $z \in \Omega$ plus one half times the number of solutions to $f(z) = w_0$ with $z \in b\Omega$ is equal to the sum

$$\frac{1}{2\pi}\sum_{j=1}^{n} \Delta_j \arg\,(f(z) - w_0),$$

which we know is equal to the sum

$$\sum \frac{1}{2}\mathcal{N}_k(0) + \sum \mathcal{N}_j(w_0)$$

where the first sum ranges over all indices k such that $|w_0| = e^{r_k}$, and the second sum ranges over indices j such that $|w_0| \neq e^{r_j}$. If $|w_0| = 1$, the only nonzero term in the sum is $\frac{1}{2}\mathcal{N}_n(0) = \frac{1}{2}$ and this shows that there is exactly one z_0 in $b\Omega$ with $f(z_0) = w_0$ and no solutions in Ω. Hence, f maps γ_n one-to-one onto the unit circle. Similarly, if $|w_0| = e^{r_1}$ we deduce that f maps γ_1 one-to-one onto the circle of radius e^{r_1}. We may now also assert that $e^{r_1} < e^{r_k} < 1$ for $k = 2, \ldots, n-1$. Suppose $2 \leq k \leq n-1$. If $|w_0| = e^{r_k}$, then the counting equation yields the following information about the number of solutions to $f(z) = w_0$ with $z \in \overline{\Omega}$. There is either exactly one solution z_0 with $z_0 \in \Omega$, or there are exactly two distinct solutions to $f(z) = w_0$ lying in $b\Omega$, or there is exactly one solution to $f(z) = w_0$ of multiplicity two in $b\Omega$. Because the net increase of arg $f(z)$ is zero around γ_k, a single valued branch of arg $f(z)$ can be defined on neighborhood of γ_k. At the points on γ_k where arg $f(z)$ assumes local maxima or minima, the mapping f must take on its values with multiplicity two. Furthermore, the counting argument above shows that there must be exactly one maximum and exactly one minimum, otherwise f would trace over sections of the circle $|w| = e^{r_k}$ more than twice. The same reasoning shows that the difference between the maximum and the minimum must be less than 2π. Finally, we may also conclude that the arc $f(\gamma_k)$ is disjoint from all the others. The proof is finished.

20

The Neumann problem in multiply connected domains

The procedure we used in Chapter 18 to solve the Neumann problem in a simply connected domain must be altered to apply in a multiply connected domain because not every harmonic function can be expressed globally as a sum $h+\overline{H}$. The modification involves the harmonic measure functions and the functions F_j' studied in Chapter 19.

Suppose φ is a harmonic function on Ω in $C^\infty(\overline{\Omega})$. We claim that there exist functions h and H in $A^\infty(\Omega)$ and constants c_j such that

$$\varphi = h + \overline{H} + \sum_{j=1}^{n-1} c_j \omega_j.$$

To see this, note that, by Lemma 19.1, it is possible to choose constants c_j so that the holomorphic function $\partial\varphi/\partial z$ is equal to the derivative h' of a function $h \in A^\infty(\Omega)$ plus $\frac{1}{2}\sum c_j F_j'$. Now $\partial/\partial z$ annihilates $\varphi - h - \sum c_j \omega_j$; so this function is an antiholomorphic function $\overline{H}$ where $H \in A^\infty(\Omega)$. This proves our claim.

The normal derivative of ω_j is easily computed. If z_0 is a point in the boundary, we may choose $\epsilon > 0$ so small that $D_\epsilon(z_0) \cap \Omega$ is a simply connected domain. On this set, we may find a harmonic conjugate for ω_j, and therefore, we may write $\omega_j = \text{Re } F_j$, where F_j is a holomorphic antiderivative of F_j'. Hence, locally, $\omega_j = \frac{1}{2}F_j + \frac{1}{2}\overline{F_j}$, and we may compute,

$$\frac{\partial \omega_j}{\partial n} = -\frac{i}{2} F_j' T + \frac{i}{2}\overline{F_j' T}.$$

But recall from Chapter 19 that $F_j' T = -\overline{F_j' T}$ on $b\Omega$. Hence, in fact,

$$\frac{\partial \omega_j}{\partial n} = -i F_j' T.$$

Using this fact, the normal derivative of φ on $b\Omega$ is seen to be equal to

$$\frac{\partial \varphi}{\partial n} = -ih'T + i\,\overline{H'T} - i\sum c_j F_j' T. \tag{20.1}$$

The next theorem shows how to solve the Neumann problem on a multiply connected domain by relating the functions h' and H' and the coefficients c_j to Szegő projections of known functions.

Theorem 20.1. *Suppose Ω is a bounded n-connected domain with C^∞ smooth boundary, and let $a \in \Omega$ be given. Suppose $\psi \in C^\infty(b\Omega)$ satisfies $\int_{b\Omega} \psi \, ds = 0$. Then, a solution to the Neumann problem*

$$\Delta\varphi = 0 \qquad \text{on } \Omega$$
$$\frac{\partial\varphi}{\partial n} = \psi \qquad \text{on } b\Omega$$

is given by

$$h + \overline{H} + \sum_{j=1}^{n-1} c_j \omega_j$$

where h and H are holomorphic functions on Ω defined via

$$h' = S_a P(\psi / \overline{L_a}\,) + \sum_{j=1}^{n-1} A_j F_j', \ \text{and}$$
$$H' = L_a P(\,\overline{\psi} / \overline{S_a}\,) + \sum_{j=1}^{n-1} B_j F_j',$$

where the constants A_j and B_j are determined by the condition that the functions on the right hand side be the derivative of a holomorphic function (i.e., by the condition that the periods of the functions on the right vanish). The constants c_j in the solution are then given by

$$c_j = -\overline{B_j} - A_j.$$

The key element in the proof of Theorem 20.1 is Lemma 18.1.

Proof. Given $\psi \in C^\infty(b\Omega)$, Lemma 18.1 yields a decomposition of ψ as $-igT + i\,\overline{GT}$ where g is in $A^\infty(\Omega)$, and G is a meromorphic function on Ω with (possibly) a simple pole at a that extends smoothly to $b\Omega$. If it is further assumed that $\int_{b\Omega} \psi \, ds = 0$, then the argument after the statement of Theorem 18.1 shows that G has no pole at a, and it follows that $G \in A^\infty(\Omega)$. In the simply connected case, we could antidifferentiate g and G to obtain the solution to the Neumann problem associated to ψ. In the multiply connected case we are studying now, although we can no longer antidifferentiate every holomorphic function, we can still use this decomposition to solve the Neumann problem. Here are the details.

The function $S_a\, P\left(\psi/\,\overline{L_a}\,\right)$ may not have a holomorphic antiderivative on Ω, but there do exist a function $h \in A^\infty(\Omega)$ and constants A_j such that

$$h' = S_a P(\psi/\,\overline{L_a}\,) + \sum_{j=1}^{n-1} A_j F_j'.$$

Similarly, there exist a function $H \in A^\infty(\Omega)$ and constants B_j such that

$$H' = L_a P(\,\overline{\psi}/\,\overline{S_a}\,) + \sum_{j=1}^{n-1} B_j F_j'.$$

Let $c_j = -\overline{B_j} - A_j$, and let $\varphi = h + \overline{H} + \sum_{j=1}^{n-1} c_j \omega_j$. The normal derivative of φ is given by (20.1). Now, using Lemma 18.1, the definitions of h and H, and the fact that $\overline{F_j' T} = -F_j' T$, we see that

$$\frac{\partial \varphi}{\partial n} = \psi - i \sum_{j=1}^{n-1} (A_j + \overline{B_j} + c_j\,) F_j' T = \psi,$$

and Theorem 20.1 is proved. □