21

The Dirichlet problem again

In this chapter, we reconsider the Dirichlet problem in a multiply connected domain. This time, we have the machinery of the harmonic measure functions developed in Chapter 19 at our disposal.

Suppose Ω is a bounded n-connected domain with C^∞ smooth boundary and suppose φ is a function in $C^\infty(b\Omega)$. Let $a \in \Omega$ be chosen so that the $n-1$ zeroes of S_a are distinct and simple. (That this can be done will be proved in Chapter 27.) Theorem 14.1 allows us to write $\varphi = h + \overline{H}$ where h is a meromorphic function on Ω that extends C^∞ smoothly up to $b\Omega$ given by $h = P(S_a\,\varphi)/S_a$ and H is in $A^\infty(\Omega)$ and is given by $H = P(L_a\,\overline{\varphi}\,)/L_a$. Notice that the set of points at which h may have poles is a subset of the set of zeroes $\{a_j\}_{j=1}^{n-1}$ of S_a.

First, we will prove that the following system can be solved for any choice of coefficients $\{B_k\}_{k=1}^{n-1}$:

$$\sum_{j=1}^{n-1} c_j P(S_a\omega_j)(a_k) = B_k, \qquad k = 1, 2, \ldots, n-1.$$

Let $A_{jk} = P(S_a\omega_j)(a_k)$. We may use identity (7.1) to compute

$$A_{jk} = \int_{b\Omega} S(a_k, z)S(z,a)\omega_j\, ds = -i\int_{\gamma_j} L(z,a_k)S(z,a)\, dz, \tag{21.1}$$

and this shows that A_{jk} is the classical period of $-iL(z,a_k)S(z,a)$ around γ_j. Now, by Theorem 19.1, the functions $L(z,a_k)S(z,a)$ and the functions F_k' span the same space. We showed in Chapter 19 that the matrix of periods associated to the F_k' is nonsingular. Hence, it follows that det $[A_{jk}] \neq 0$ and our system can be solved.

Let $\varphi \in C^\infty(b\Omega)$ be given and set $B_k = P(S_a\varphi)(a_k)$. Define $\psi = \varphi - \sum_{j=1}^{n-1} c_j\omega_j$ where the c_j satisfy the linear system,

$$\sum_{j=1}^{n-1} A_{jk}c_j = B_k, \qquad k = 1, 2, \ldots, n-1.$$

Now define

$$h = \frac{P(S_a\,\psi)}{S_a}$$

and

$$H = \frac{P(L_a\,\overline{\psi})}{L_a}.$$

The linear system was conceived so that $P(S_a\psi)$ has a zero at each of the zeroes of S_a. Since the zeroes of S_a are simple zeroes, this means that h, which is the quotient of these two functions, has no poles in Ω. Thus h and H are both in $A^\infty(\Omega)$ and it follows from Theorem 14.1 that $h+\overline{H}$ gives the harmonic extension of ψ to Ω. Hence, the harmonic extension of φ is given by $h+\overline{H}+\sum_{j=1}^{n-1} c_j\omega_j$. We summarize this result in the following theorem.

Theorem 21.1. *Let Ω be a bounded n-connected domain with C^∞ smooth boundary and let A_{jk} denote the matrix of periods for the functions $-iL(z,a_k)S(z,a)$ as given by (21.1). Given $\varphi \in C^\infty(b\Omega)$, let c_j solve the system*

$$\sum_{j=1}^{n-1} A_{jk}c_j = P(S_a\varphi)(a_k), \qquad k = 1,2,\ldots,n-1.$$

The harmonic extension of φ to Ω is given by

$$h+\overline{H}+\sum_{j=1}^{n-1} c_j\omega_j,$$

where, using the notation $\psi = \varphi - \sum c_j\omega_j$, h and H are functions in $A^\infty(\Omega)$ given by

$$h = \frac{P(S_a\,\psi)}{S_a} \qquad \text{and}$$
$$H = \frac{P(L_a\,\overline{\psi})}{L_a}.$$

22

Area quadrature domains

In the next two chapters, we hope to convey the beauty of the subject of quadrature domains in the plane, and the utility of the tools and techniques described in the previous chapters to understand quadrature domains. To make these chapters self-contained, we do not go into the utmost generality. For the whole story, see the Bibliographic Notes on page 200.

The unit disc is the most famous of area quadrature domains. When a holomorphic function is averaged over the disc with respect to area measure, the value of the function at the origin is obtained. More generally, a domain Ω in the plane of finite area is called an *area quadrature domain* if the average of a holomorphic function in the Bergman space over the domain with respect to area measure is a finite linear combination of values of the function and its derivatives at finitely many points in Ω. The points and the coefficients are fixed in this "quadrature identity" and the identity holds for all holomorphic functions in the Bergman space. In other words, there exist points $\{z_j\}_{j=1}^N$ in Ω, complex constants c_{jk}, and nonnegative integers m_j such that

$$\iint_\Omega h\ dA = \sum_{j=1}^{N}\sum_{k=0}^{m_j} c_{jk}h^{(k)}(z_j) \tag{22.1}$$

for all h in $H^2(\Omega)$. In this chapter, we will show that area quadrature domains share many of the properties of the unit disc and that they turn out to be very abundant. We will also explain how to view our results as an improvement upon the Riemann mapping theorem.

Before we begin, we must define higher order versions of the Bergman kernel. In Chapter 15, we showed that the averaging property of holomorphic functions allowed us to the write the Bergman kernel K_a as the Bergman projection of a function φ_a in $C_0^\infty(\Omega)$. We now take a closer look at this fact with an eye to differentiating it with respect to a. Let θ be a real valued radially symmetric C^∞ function compactly supported in the unit disc such that $\iint \theta\, dA = 1$. Fix a point a_0 in Ω and let $\epsilon > 0$ be less than the distance from a_0 to the boundary of Ω. For a near a_0, the function φ_a that we described in Chapter 15, may be written

$$\varphi_a(z) = \frac{1}{\epsilon^2}\theta\left(\frac{z-a}{\epsilon}\right).$$

Recall that

$$h(a) = \langle h, \varphi_a \rangle_\Omega \tag{22.2}$$

for all $h \in H^2(\Omega)$ and that is why we were able to conclude that $K_a = B\varphi_a$. Let φ_a^m be defined to be $(\partial^m/\partial \bar{a}^m)\varphi_a$. Note that this function is in $C_0^\infty(\Omega)$ and that it is the conjugate of $(\partial^m/\partial a^m)\varphi_a$ because φ_a is real valued. If a is close to a_0, we may differentiate (22.2) m times with respect to a to obtain

$$h^{(m)}(a) = \langle h, \varphi_a^m \rangle_\Omega.$$

Let $K_a^m = B\varphi_a^m$. This function represents the functional $h \mapsto h^{(m)}(a)$ in the sense that

$$h^{(m)}(a) = \langle h, K_a^m \rangle_\Omega.$$

Let K_a^0 denote the Bergman kernel K_a. The *Bergman span* of Ω is defined to be the space of all linear combinations of functions of the form K_a^m as a ranges over Ω and m ranges over nonnegative integers. We may equate $\langle K_a^m, K_z \rangle_\Omega$ with the conjugate of $\langle K_z, K_a^m \rangle_\Omega$ to deduce that

$$K_a^m(z) = \frac{\partial^m}{\partial \bar{a}^m} K(z, a).$$

We remark here that equating $\langle K_b^n, K_a^m \rangle_\Omega$ with the conjugate of $\langle K_a^m, K_b^n \rangle_\Omega$ allows one to express all the derivatives of the Bergman kernel in both variables in terms of integrals that are easily bounded on compact sets via the basic estimate proved in Chapter 15. In this way, it can be deduced that $K(z, w)$ is C^∞ smooth in z and w in $\Omega \times \Omega$, holomorphic in z, and antiholomorphic in w.

We now turn to the study of area quadrature domains. Assume that Ω is an area quadrature domain. Note that the function that is identically one is in the Bergman space because of the finite area assumption, and if h is another function in the Bergman space, then

$$\iint_\Omega h \; dA = \langle h, 1 \rangle_\Omega.$$

Thus, we see that h is in $L^1(\Omega)$ by the Cauchy-Schwarz inequality, and so the integral in the quadrature identity for Ω is well defined. Let Q denote the following linear combination of functions $K_{z_j}^k$ in the Bergman span inspired by the quadrature identity (22.1) that we assume holds:

$$Q(z) = \sum_{j=1}^{N} \sum_{k=0}^{m_j} \overline{c_{jk}} K_{z_j}^k. \tag{22.3}$$

Notice that

$$\langle h, 1\rangle_\Omega = \langle h, Q\rangle_\Omega$$

for $h \in H^2(\Omega)$ because both sides yield the quadrature identity in h. Since the functions 1 and Q are both in the Bergman space, and since h is arbitrary, if must be that $Q \equiv 1$. The converse of this fact is also clearly true, namely that if the function 1 is in the Bergman span, then Ω is an area quadrature domain.

Theorem 22.1. *A domain of finite area is an area quadrature domain if and only if the function that is identically equal to one is in the Bergman span.*

We will now combine this theorem with the transformation properties of the Bergman kernel functions under biholomorphic mappings to obtain the following theorem.

Theorem 22.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between domains of finite area. Then Ω_2 is an area quadrature domain if and only if f' belongs to the Bergman span of Ω_1.*

Proof. Suppose f is a biholomorphic mapping as in the statement of the theorem. Because $|f'|^2$ is equal to the absolute value of the *real* Jacobian determinate of the mapping as a mapping from $\mathbb{R}^2$ into itself, it follows from the classical change of variables formula (see Chapter 15) that

$$\iint_{\Omega_2} h\ dA = \langle h, 1\rangle_{\Omega_2} = \iint_{\Omega_1} |f'|^2 (h \circ f)\ dA = \langle f'(h \circ f), f'\rangle_{\Omega_1}.$$

If f' is in the Bergman span, then the last inner product yields a linear combination of values and derivatives of $f'(h \circ f)$ at finitely many points in Ω_1, which is a fixed finite linear combination of values and derivatives of h at finitely many points in Ω_2. Consequently, Ω_2 is an area quadrature domain.

Conversely, suppose that Ω_2 is an area quadrature domain. Let $F = f^{-1}$. If h is a function in $H^2(\Omega_1)$, then identity (15.1) yields

$$\langle h, f'\rangle_{\Omega_1} = \langle F'(h \circ F), 1\rangle_{\Omega_2} = \iint_{\Omega_2} F'(h \circ F)\ dA,$$

and this last integral is equal to a linear combination of values and derivatives of $F'(h \circ F)$ by virtue of the quadrature identity for Ω_2. Such a linear combination is a linear combination of values and derivatives of h at finitely many points in Ω_1. There is a function Q in the Bergman span of Ω_1 that yields the same linear combination when paired with h. Since $\langle h, f'\rangle_{\Omega_1} = \langle h, Q\rangle_{\Omega_1}$ holds for all h in the Bergman space, it follows that $f' = Q$, i.e., that f' is in the Bergman span. $\square$

When Theorem 22.2 is combined with the Riemann mapping theorem and the known formula for the Bergman kernel on the disc, the following lovely result is obtained.

Theorem 22.3. *A simply connected domain of finite area is an area quadrature domain if and only if the inverse of a Riemann mapping function is rational and without poles on the boundary.*

Proof. Suppose that Ω is a simply connected domain of finite area and let $f : \Omega \to D_1(0)$ be a Riemann mapping function associated to a point in Ω. Recall that the Bergman kernel for the unit disc is $\pi^{-1}/(1-z\bar{w})^2$. Let $F = f^{-1}$ be the inverse of the Riemann map. Theorem 22.2 yields that Ω is an area quadrature domain if and only if F' is in the Bergman span of the disc. It is easy to verify that the Bergman span of the disc is exactly the set of rational functions with residue free poles outside the closed unit disc. (Note that the complex polynomials belong to this space.) Such rational functions have rational antiderivatives and the set of all such antiderivatives is the set of rational functions with poles outside the closed unit disc. □

We remark that the reasoning in the proof of Theorem 22.3 can be used to give a quick proof that discs are the only "one point" simply connected quadrature domains. Indeed, if Ω is a simply connected area quadrature domain satisfying

$$\iint_\Omega h \, dA = ch(a)$$

for all h in the Bergman space, let f be a Riemann map that takes a to the origin and let $K(z, w)$ denote the Bergman kernel associated to Ω. Putting the function $h \equiv 1$ in the quadrature identity shows that c is equal to the area of Ω. The proof of Theorem 22.1 yields that K_a is equal to the constant $C = 1/c$. The transformation formula for the Bergman kernels yields

$$F'(z)K(F(z), a) = \frac{1}{\pi(1 - z\,\overline{f(a)}\,)^2}\overline{f'(a)}, \tag{22.4}$$

and since $f(a) = 0$ and $K(\cdot, a)$ is constant, we conclude that $F'(z)$ is a constant. Consequently $F(z)$ is complex linear and Ω is a disc.

It is interesting to next consider simply connected domains in the plane with two point quadrature identities. Similar reasoning can be used, mapping one of the points a_1 in the quadrature identity to the origin via a Riemann map and using the transformation formula, to show that such domains must be the famous Neumann ovals. Indeed, the two point quadrature identity yields constants c_1 and c_2 such that $c_1K(z,a_1)+c_2K(z,a_2)\equiv 1$. Adding up the appropriate linear combinations of equation (22.4) yields

$$F'(z) = A_1 + \frac{A_2}{(1 - z\,\overline{f(a_2)}\,)^2},$$

where A_1 and A_2 are constants. It follows that $F(z) = C_0 + A_1z + B_2/(z-b)$, where C_0, A_1, B_2 are complex constants and $b = 1/\overline{f(a_2)}$ is a point outside the closed unit disc. If F is one-to-one on the unit disc, then $F(D_1(0))$ is indeed a two point area quadrature domain. See Shapiro [Sh] for more about these special domains. We will soon see that continuing this line of thought leads to a class of quadrature domains that are "dense" in the realm of simply connected domains with smooth boundaries.

Theorem 22.3 reveals that simply connected area quadrature domains have algebraic Riemann mapping functions and particularly nice boundaries. In particular, the function F in the proof is holomorphic in a neighborhood of the closed unit disc. If F' is nonvanishing on the closed unit disc, then it follows that the boundary of the simply connected area quadrature domain must be C^∞ smooth and real analytic (and, in fact, real algebraic). If F' has a zero on the unit circle, then near such a zero, for F to map one-to-one onto a domain, the zero can be at most a simple zero and F maps the zero to a boundary point of Ω that is the terminal end of a cusp pointing into the domain. (Otherwise, the local mapping theorem would show that F is not one-to-one on a small disc about the boundary point intersected with the unit disc.) Thus, Ω either has C^∞ smooth boundary, or is C^∞ smooth except at finitely many inward pointing cusps.

We will now show that simply connected area quadrature domains without cusps in the boundary are "dense" in the realm of simply connected domains with smooth boundary. Suppose that Ω is a simply connected domain bounded by a C^∞ smooth curve. Let f denote a Riemann map associated to a point in the domain. We know that f is in $C^\infty(\overline{\Omega})$ and that the inverse F of f is in $C^\infty(\overline{D_1(0)})$. If $\rho < 1$, then $F(\rho z)$ is holomorphic in a neighborhood of the closed unit disc and we may approximate $F(\rho z)$ in $C^\infty(\overline{D_1(0)})$ by a Taylor polynomial $P(z)$. By taking ρ to be sufficiently close to one and taking the polynomial to be sufficiently close to $F(\rho z)$ in $C^\infty(\overline{D_1(0)})$, we obtain a domain $P(D_1(0))$ that is an area quadrature domain without cusps that is as close to Ω in C^∞ as desired in the sense that the biholomorphic map $P \circ f$ is as close to the identity in $C^\infty(\overline{\Omega})$ as desired. This is the improvement upon the Riemann mapping theorem alluded to above. Instead of mapping the domain to the unit disc, which might be a far away quadrature domain, we have mapped to a nearby quadrature domain. Since quadrature domains have many properties in common with the unit disc, this nearby quadrature domain might be more useful than a far away disc in numerical computations.

The argument above can be reworked on a domain bounded by a Jordan curve to obtain a mapping that is uniformly close to the identity map and that maps to an area quadrature domain without cusps in the boundary. The key to the argument is Carathéodory's theorem about continuous extension to the boundary of Riemann maps in this setting. This yields Gustafsson's theorem about the *uniform* density of area quadrature domains among Jordan domains.

We now reveal one of the very special properties of area quadrature domains that make them akin to the unit disc. Simply connected area quadrature domains without cusps in the boundary have a well behaved "Schwarz function" that is quite useful. The Schwarz function for the unit disc is $S(z) = 1/z$. Notice that it is meromorphic on the disc, it is holomorphic on a neighborhood of the unit circle, and $S(z) = \bar{z}$ on the unit circle. We remark here that the antiholomorphic reflection function studied in Chapter 11 is given by $R(z) = \overline{S(z)} = 1/\bar{z}$. To motivate what we are about to do next, keep in mind that we would expect antiholomorphic reflection functions to be invariant under biholomorphic mappings that extend holomorphically past the boundaries.

Suppose that Ω is a simply connected area quadrature without cusps and let f be a Riemann map with inverse F as above. Since F is rational, we may think of F as defined on the whole complex plane minus finitely many poles. The *Schwarz function* $S(z)$ for Ω is defined via

$$S(z) = \overline{F(1/\overline{f(z)}\,)}.$$

Note that S is holomorphic where it is well defined because a composition of antiholomorphic functions is holomorphic. Also note that S has no singularities on the unit circle and at worst poles where f has zeroes or where $1/\overline{f}$ maps to a pole of F. We claim that $S(z)$ is such that $S(z) = \bar{z}$ on the boundary of Ω and S extends meromorphically to Ω. Indeed, if z is a boundary point of Ω, then $f(z)$ is a point on the unit circle, and so $f(z) = 1/\overline{f(z)}$. Since F is the inverse of f, it follows that $S(z) = \bar{z}$. Also, $1/f$ maps Ω onto-to-one onto the outside of the closed unit disc union the point at infinity. Since F is rational, we conclude that $S(z)$ is meromorphic on Ω. Since $S(z)$ has no singularities on the boundary, $S(z)$ extends to be holomorphic on a neighborhood of the boundary.

We now turn to the study of area quadrature in the multiply connected category. Since the Riemann mapping theorem is no longer at our disposal, we will have to come up with new techniques, but we will obtain many similar results to those in the simply connected setting.

Suppose Ω_1 is a finitely connected domain of finite area. The same argument used in the proof of Lemma 12.1 (see also Ahlfors [Ah, p. 252]) yields that there exists a biholomorphic mapping $f : \Omega_1 \to \Omega_2$ to a bounded domain Ω_2 with smooth real analytic boundary. Let $F = f^{-1}$ denote the inverse of f. Theorem 22.2 states that Ω_1 is an area quadrature domain if and only if F' is in the Bergman span of Ω_2. Note that Theorem 17.1 together with the fact that K_a^m is the Bergman projection of a function in $C_0^\infty(\overline{\Omega})$ shows that the Bergman span of a bounded domain with smooth real analytic boundary consists of functions that extend holomorphically past the boundary. Thus, we have proved the following theorem.

Theorem 22.4. *If Ω is a finitely connected area quadrature domain, then Ω is a bounded domain bounded by finitely many nonintersecting real analytic curves that are either C^∞ smooth or C^∞ smooth except at finitely many inward pointing cusps.*

Suppose that Ω is a finitely connected area quadrature domain without cusps in the boundary. We will now show that Ω has a Schwarz function that extends meromorphically to Ω exactly as in the simply connected case. Since Ω has C^∞ smooth boundary, we may apply the improved Cauchy integral formula of Theorem 2.1 to the function $u(z) = \bar{z}$ to obtain

$$\bar{z} = \mathcal{C}\bar{z} + \frac{1}{2\pi i}\iint_{w\in\Omega} \frac{1}{w-z} dw \wedge d\bar{w}$$

for $z \in \Omega$. Recall that the boundary values of the integral over Ω can be evaluated by letting z approach the boundary from the *outside* of the domain (see Theorem 17.2). If z is outside the closure of the domain, then the quadrature identity for Ω yields a rational function $R(z)$ in z with no poles outside Ω. Thus, we conclude that $\bar{z} = h(z) + R(z)$ on the boundary, where $h = \mathcal{C}\bar{z}$ is a holomorphic function in $C^\infty(\overline{\Omega})$ and $R(z)$ is a rational function without poles outside Ω. The Schwarz function is therefore given by $h + R$ and is seen to extend meromorphically to Ω.

To illustrate the utility of the Schwarz function, we will use it to prove the following result.

Theorem 22.5. *Suppose that Ω is a finitely connected area quadrature domain without cusps in the boundary. Then the complex polynomials belong to the Bergman span of Ω.*

Proof. Suppose that Ω is a domain as in the statement of the theorem. Let $S(z)$ denote the Schwarz function for Ω. Given a function

$h \in A^\infty(\Omega)$, observe that the complex Green's identity yields

$$\begin{aligned}\langle h, z^n\rangle_\Omega &= \frac{(-i/2)}{n+1}\iint_\Omega \frac{\partial}{\partial \bar z}\left[h(z)\bar z^{n+1}\right] d\bar z \wedge dz \\ &= -\frac{(i/2)}{n+1}\int_{b\Omega} h(z)\bar z^{n+1}\ dz \\ &= -\frac{i/2}{n+1}\int_{b\Omega} h(z)S(z)^{n+1}\ dz. \end{aligned} \tag{22.5}$$

This last integral is equal to a finite linear combination of values and derivatives of h at finitely many points via the residue theorem. There is a function q in the Bergman span that produces this same linear combination when paired with such an h. Since $A^\infty(\Omega)$ is dense in $H^2(\Omega)$, we conclude that $z^n = q(z)$, i.e., that z^n is in the Bergman span. Since this argument works for any nonnegative integer n, we conclude that all complex polynomials are in the Bergman span. □

Since the function 1 being in the Bergman span is equivalent to the area quadrature condition, one can say that a smooth finitely connected domain of finite area is an area quadrature domain if and only if the Bergman span contains all complex polynomials.

We remark that letting $n = 0$ in equation (22.5) shows that if a bounded domain Ω with C^∞ smooth boundary has a Schwarz function $S(z)$ that is meromorphic on Ω, extends continuously to the boundary, and satisfies $S(z) = \bar z$ on the boundary, then Ω is a smooth area quadrature domain. Thus, having a Schwarz function like this is equivalent to being an area quadrature in the realm of smooth domains. Note that the Schwarz function has poles at the points that appear in the quadrature identity and the numbers $m_j + 1$ are the orders of the poles. It can be shown that the existence of a Schwarz function with the properties we have been using is equivalent to the area quadrature condition in much more generality and we direct the reader to the Bibliographic Notes for avenues to explore.

A theorem that is closely related to the fact that one-point quadrature domains are discs states that the only domains with complex rational Schwarz functions are discs. Indeed, if $S(z)$ is a rational Schwarz function for a domain, then $R(z) = \overline{S(z)}$ is the antiholomorphic reflection function studied in Chapter 11. The fact that $S(z) = \bar z$ on the boundary yields that $R(z) = z$ on the boundary. Consequently, $R \circ R$ is holomorphic near the boundary and equal to z on the boundary. Therefore, $R(R(z)) = z$ globally, i.e., R is equal to its own inverse. The only rational functions of $\bar z$ with this property are the linear fractional transformations. We conclude that $S(z)$ has at most one pole (which must fall inside the domain), and therefore equation (22.5) with $n = 0$ shows that the domain is a one-point quadrature domain, and hence, a disc.

We remark that the Bergman kernel and the Schwarz function are highly related in area quadrature domains, and that will be the subject of the last chapter of this book.

We will close this chapter by sketching a proof that any bounded domain with C^∞ smooth boundary can be approximated by finitely connected area quadrature domains without cusps via a biholomorphic mapping that is as C^∞-close to the identity as desired.

We have already handled the simply connected case, so suppose that Ω is a bounded multiply connected domain with C^∞ smooth boundary. We will prove in Chapter 30 that the space of all linear combinations of functions of the form K_a as a ranges over Ω is dense in $A^\infty(\Omega)$, and hence the Bergman span of Ω is dense in $A^\infty(\Omega)$. We now approximate the function $H(z) \equiv 1$ in $A^\infty(\Omega)$ by a function Q in the Bergman span. Suppose that Ω has n boundary curves γ_j, $j = 1, \ldots, n$. Let the first $n-1$ of those curves denote inner boundary curves and let b_j, $j = 1, \ldots, n-1$ be points inside the bounded domains determined by the inner boundary curves, one per curve. We may also approximate the functions $h_j(z) = 1/(z - b_j)$ in $A^\infty(\Omega)$ by functions q_j in the Bergman span. Note that the period matrix $\int_{\gamma_i} h_j \, dz$ as i and j range from 1 to $n-1$ is nonsingular. Hence, by making our approximations close enough, we may assume that the period matrix of the functions q_j is also nonsingular. Since Q is close to one, the periods of Q are small. Hence, by taking Q sufficiently close to one, we may assume that there are small constants ϵ_j such that the periods of $Q - \sum_{j=1}^{n-1} \epsilon_j q_j$ are *zero*. Because the periods are zero, we know there is an antiderivative f of this last function on Ω. If we make all our approximations sufficiently close, we see that f can be made arbitrarily close to the identity map (since f' is close to one). In particular, we may assume that f is one-to-one on Ω and close to the function z in $C^\infty(\overline{\Omega})$. Since f' is in the Bergman span, the image of Ω under f is an n-connected area quadrature domain without cusps in the boundary that is as C^∞ close to Ω as desired. This result can clearly be viewed as an improvement upon the Riemann mapping theorem, which does not hold in the multiply connected domain setting.

To show that a domain bounded by finitely many nonintersecting Jordan curves can be approximated by a smooth area quadrature domain (Gustafsson's theorem [Gu1]), first map the domain to a domain contained in the unit disc via a Riemann map f associated to the simply connected domain given by the inside of the outer boundary curve. If F is the inverse of this map, we may use the dilated map $F_\rho(z) = F(\rho z)$ as we did in the simply connected case to obtain a map $F_\rho \circ f$ that is close to the identity map that maps the domain to a nearby domain bounded by Jordan curves, where the outer boundary is now real analytic. If b is a point inside one of the bounded domains determined by an inner boundary curve of this new domain, we may invert the domain using the map $1/(z-b)$ to send that inner boundary curve to the outer boundary. We now repeat the process for this new outer boundary, noting that the boundary curve represented by the old outer boundary is now a real analytic curve, too. Hence, two of the boundary curves are real analytic curves at the end of this next iteration. We now compose with the inverse of $1/(z-b)$ (which is $b+(1/w)$) to obtain a domain close to the original domain where two of the boundary curves are real analytic. It is now clear how to continue this process until a nearby domain with real analytic boundary is obtained. Finally, we map this domain with real analytic boundary to a quadrature domain via a map that is C^∞ close to the identity and compose to obtain a map to a quadrature domain that is uniformly close to the identity.

23

Arc length quadrature domains

Not only is the unit disc the most famous of area quadrature domains, it is also the most famous of boundary arc length quadrature domains. When a holomorphic function that extends continuously to the boundary is averaged over the unit circle with respect to arc length measure, the value of the function at the origin is obtained. More generally, a finitely connected domain in the plane bounded by n nonintersecting C^1 smooth curves is called an *arc length quadrature domain* if the average of a holomorphic function that extends continuously to the closure of the domain over the boundary with respect to arc length measure is a finite linear combination of values of the function and its derivatives at finitely many points in the domain. The points and the coefficients are fixed in this quadrature identity even though the holomorphic function is allowed to vary. In this chapter, we will restrict our attention to arc length quadrature domains with C^∞ smooth boundaries, and we will call such domains *smooth arc length quadrature domains*. In this context, a bounded domain Ω with C^∞ smooth boundary is called a smooth arc length quadrature domain if there exist points $\{z_j\}_{j=1}^N$ in Ω, complex constants c_{jk}, and nonnegative integers m_j such that

$$\int_{b\Omega} h\ ds = \sum_{j=1}^{N} \sum_{k=0}^{m_j} c_{jk} h^{(k)}(z_j) \tag{23.1}$$

for all h in the Hardy space $H^2(b\Omega)$. In this chapter, we will show that arc length quadrature domains are to the Szegő kernel as area quadrature domains are to the Bergman kernel. We will also consider domains that are like the unit disc in that they are *both* area and arc length quadrature domains. Once again, these results can be viewed as improvements upon the Riemann mapping theorem.

The reader will notice that this chapter is highly parallel to the previous one.

To begin, we must define higher order versions of the Szegő kernel much like we did in Chapter 22 for the Bergman kernel. In Chapter 11, we showed that the Szegő kernel S_a was the Szegő projection of the

Cauchy kernel C_a. We can differentiate the identity

$$h(z) = \langle h, C_a \rangle_b = \langle h, S_a \rangle_b$$

exactly as we did for the Bergman kernel to obtain higher order functions S_a^m which are Szegő projections of $(\partial^m/\partial \bar{a}^m)C_a$. Analogous arguments show that $S_a^m(z) = (\partial^m/\partial \bar{a}^m)S(z,a)$. Note that S_a^m is in $A^\infty(\Omega)$ and

$$h^{(m)}(a) = \langle h, S_a^m \rangle_b.$$

Let S_a^0 denote S_a.

The *Szegő span* of Ω is defined to be the space of all linear combinations of functions of the form S_a^m as a ranges over Ω and m ranges over nonnegative integers. Note that Theorem 9.1 yields that the Szegő span is dense in $A^\infty(\Omega)$ in smooth domains.

We now show that there is a theorem analogous to Theorem 22.1 for smooth arc length quadrature domains.

Assume that Ω is a smooth arc length quadrature domain. Note that if h is in the Hardy space, then

$$\int_\Omega h \, ds = \langle h, 1 \rangle_b.$$

We assume that the quadrature identity (23.1) holds. Let Q denote the following linear combination of functions $S_{z_j}^k$ in the Szegő span:

$$Q(z) = \sum_{j=1}^{N} \sum_{k=0}^{m_j} \overline{c_{jk}} S_{z_j}^k. \tag{23.2}$$

Notice that

$$\langle h, 1 \rangle_b = \langle h, Q \rangle_b$$

for $h \in H^2(b\Omega)$ because both sides yield the quadrature identity in h. Since the functions 1 and Q are both in the Hardy space, and since h is arbitrary, if must be that $Q \equiv 1$. The converse of this fact is also clearly true, namely that if Ω is a bounded domain with C^∞ smooth boundary and the function 1 is in the Szegő span, then Ω is a smooth arc length quadrature domain. Thus, we have proved the following theorem.

Theorem 23.1. *A bounded domain with C^∞ smooth boundary is a smooth arc length quadrature domain if and only if the function that is identically equal to one is in the Szegő span.*

We can combine this theorem with transformation properties of the Szegő kernel functions under biholomorphic mappings to obtain the following theorem analogous to Theorem 22.2.

Theorem 23.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded domains with C^∞ smooth boundaries. Then Ω_2 is a smooth arc length quadrature domain if and only if $\sqrt{f'}$ belongs to the Szegő span of Ω_1.*

Proof. Suppose f is a biholomorphic mapping as in the statement of the theorem. Theorem 12.1 informs us that f extends nicely to the boundary and there is a holomorphic square root of f' that also extends nicely. Let $\sqrt{f'}$ denote one of the two possible square root functions. If h is in $H^2(b\Omega_2)$, it follows from the discussion before equation (12.3) that

$$\int_{b\Omega_2} h\ ds = \langle h, 1\rangle_{b\Omega_2} = \iint_{b\Omega_1} |f'|(h\circ f)\ dA = \langle \sqrt{f'}(h\circ f), \sqrt{f'}\rangle_{b\Omega_1}.$$

If $\sqrt{f'}$ is in the Szegő span, then the last inner product yields a linear combination of values and derivatives of $\sqrt{f'}(h\circ f)$ at finitely many points in Ω_1, which is a fixed finite linear combination of values and derivatives of h at finitely many points in Ω_2. Consequently, Ω_2 is a smooth arc length quadrature domain.

Conversely, suppose that Ω_2 is an arc length quadrature domain. Let $F = f^{-1}$. If h is a function in $H^2(b\Omega_1)$, then identity (12.4) yields

$$\langle h, \sqrt{f'}\rangle_{b\Omega_1} = \langle \sqrt{F'}(h\circ F), 1\rangle_{b\Omega_2} = \int_{b\Omega_2} \sqrt{F'}(h\circ F)\ ds,$$

and this last integral is equal to a linear combination of values and derivatives of $\sqrt{F'}(h\circ F)$ by virtue of the quadrature identity for Ω_2. Such a linear combination is a linear combination of values and derivatives of h at finitely many points in Ω_1. There is a function Q in the Szegő span of Ω_1 that yields the same linear combination when paired with h. Since $\langle h, \sqrt{f'}\rangle_{b\Omega_1} = \langle h, Q\rangle_{b\Omega_1}$ holds for all h in the Hardy space, it follows that $\sqrt{f'} = Q$, i.e., that $\sqrt{f'}$ is in the Szegő span. □

When Theorem 23.2 is combined with the Riemann mapping theorem and the known formula for the Szegő kernel on the disc, the following analogue to Theorem 22.3 is obtained.

Theorem 23.3. *A bounded simply connected domain with C^∞ smooth boundary is an arc length quadrature domain if and only if the derivative of the inverse of a Riemann mapping function is the square of a rational function without poles on the closed unit disc.*

Proof. Suppose that Ω is a bounded simply connected domain with C^∞ smooth boundary and let $f : \Omega \to D_1(0)$ be a Riemann mapping function associated to a point in Ω. Recall that the Szegő kernel for the unit disc is $(2\pi)^{-1}/(1 - z\bar{w})$. Let $F = f^{-1}$ be the inverse of the Riemann map.

Theorem 23.2 yields that Ω is an area quadrature domain if and only if $\sqrt{F'}$ is in the Szegő span of the disc. It is easy to verify that the Szegő span of the disc is exactly the set of rational functions with poles outside the closed unit disc. (Note that the complex polynomials belong to this space.) Hence, $\sqrt{F'}$ is in the Szegő span if and only if F' is the square of such a function and the proof is complete. $\square$

We remark here that if Ω is a one-point simply connected smooth arc length quadrature domain, then we can adapt the proof of Theorem 23.3 just as we did after the proof of Theorem 22.3 to deduce that Ω must be a disc. Indeed, the argument shows that the square root of the derivative of the inverse of a Riemann map that maps the one point to the origin is constant, and so the inverse is complex linear and the domain is a disc.

Theorem 23.3 reveals that simply connected smooth arc length quadrature domains have very nice boundaries. Indeed, the function F in the proof is holomorphic in a neighborhood of the closed unit disc and F' must be nonvanishing on the closed unit disc for it to have a square root that is smooth up to the boundary. It therefore follows that the boundary of the simply connected arc length quadrature domain must be C^∞ smooth and real analytic.

We will now show that smooth simply connected arc length quadrature domains are dense in the realm of simply connected domains with smooth boundary. Suppose that Ω is a simply connected domain bounded by a C^∞ smooth curve. Let f denote a Riemann map associated to a point in the domain, which we know is in $C^\infty(\overline{\Omega})$. The inverse F of f is in $C^\infty(\overline{D_1(0)})$. If $\rho < 1$, let $F_\rho(z) = F(\rho z)$. Notice that F_ρ is holomorphic in a neighborhood of the closed unit disc. There is a holomorphic square root of $\sqrt{F'_\rho}$ that also extends smoothly to the closure of the unit disc. We may approximate $\sqrt{F'_\rho}$ in $C^\infty(\overline{D_1(0)})$ by a Taylor polynomial $q(z)$. Let $P(z)$ be a polynomial that is an antiderivative of $q(z)^2$. We may choose the constant of integration so that $P'(z)$ is close to F'_ρ in $C^\infty(\overline{D_1(0)})$. By taking ρ to be sufficiently close to one and taking the polynomial approximates sufficiently close to their targets, we obtain a domain $P(D_1(0))$ that is *both* a smooth arc length quadrature and an area quadrature domain without cusps that is as close to Ω in C^∞ as desired in the sense that the biholomorphic map $P \circ f$ is as close to the identity in $C^\infty(\overline{\Omega})$ as desired. This is an improvement upon the improvement upon the Riemann mapping theorem of the last chapter. We call smooth domains that are both area and arc length quadrature domains *double* quadrature domains. Instead of mapping the domain to the unit disc, which might be a far away double quadrature domain, we have mapped to a nearby double quadrature domain. Double quadrature domains have even more properties in common with the unit disc.

We now turn to the study of arc length quadrature domains in the more general finitely connected setting.

First, we show that smooth arc length quadrature domains have the very special property that their complex unit tangent vector functions $T(z)$ extend meromorphically to the domain. Indeed, if we differentiate identity 7.1 with respect to a and add up a linear combination of conjugates of functions in the Szegő span that sum to one (given by Theorem 23.1), we obtain that $1 = HT$ on the boundary, where $H(z)$ is a meromorphic function on Ω that extends smoothly to the boundary and is given by a linear combination of the Garabedian kernel $L_a(z)$ and its derivatives in a at finitely many points a in Ω. Hence $1/H$ is the meromorphic extension we seek. Note that, since $\overline{T} = 1/T$, it follows that $\overline{T}$ extends as the meromorphic function H to Ω. The following computation shows that this condition of extendibility of T is equivalent to the arc length quadrature condition in the realm of smooth domains. Indeed, if Ω is a bounded C^∞ smooth domain and h is in $A^\infty(\Omega)$, then

$$\int_{b\Omega} h\ ds = \int_{b\Omega} h\ \overline{T}\,T\ ds = \int_{b\Omega} h\ H\,dz,$$

where H is the meromorphic extension of $\overline{T}$, and this last integral is a linear combination of values and derivatives of h at finitely many points in Ω via the residue theorem. Since $A^\infty(\Omega)$ is dense in the Hardy space, this identity extends to $H^2(b\Omega)$ and we conclude that Ω is an arc length quadrature domain.

It can be shown that the extension property of T is equivalent to the arc length quadrature condition in much more generality and, once again, we direct the curious reader to the Bibliographic Notes.

We next show that the boundaries of smooth arc length quadrature domains are particularly nice. Suppose Ω_1 is a bounded domain with C^∞ smooth boundary. We know that there exists a biholomorphic mapping $f : \Omega_1 \to \Omega_2$ to a bounded domain Ω_2 with smooth real analytic boundary. Let $F = f^{-1}$ denote the inverse of f. Theorem 23.2 states that Ω_1 is an area quadrature domain if and only if $\sqrt{F'}$ is in the Szegő span of Ω_2. Theorem 11.2 can be used to show that the Szegő span of a bounded domain with smooth real analytic boundary consists of functions that extend to be holomorphic in a neighborhood of the closure of the domain. Thus, $\sqrt{F'}$ extends holomorphically past the boundary, and consequently, so does F' and F. As in the area quadrature domain case, F' can vanish to at most first order on the boundary in order to map one-to-one to a domain, but since $\sqrt{F'}$ extends, F' cannot vanish on the boundary to first order. We have proved the following theorem.

Theorem 23.4. *If Ω is a finitely connected smooth arc length quadrature* *domain, then Ω is a bounded domain bounded by finitely many nonintersecting C^∞ smooth real analytic curves.*

In the last chapter, we saw that being an area quadrature domain was essentially equivalent to the complex polynomials belonging to the Bergman span. The analogous theorem for arc length quadrature domains is as follows.

Theorem 23.5. *Suppose that Ω is a bounded domain with C^∞ smooth boundary. Then the complex polynomials belong to the Szegő span of Ω if and only if Ω is a smooth double quadrature domain.*

Proof. Suppose that Ω is a smooth double quadrature domain. Let $S(z)$ denote the Schwarz function for Ω and let H denote the meromorphic extension of $\overline{T}$ to Ω. Given a function $h \in A^\infty(\Omega)$, observe that

$$\langle h, z^n \rangle_b = \int_{b\Omega} h\, \bar{z}^n\, \overline{T}\, T\ ds = \int_{b\Omega} h\, S(z)^n H(z)\ dz,$$

which is a finite linear combination of values and derivatives of h at finitely many points via the residue theorem. There is a function q in the Szegő span that produces this same linear combination when paired with such an h. Since $A^\infty(\Omega)$ is dense in $H^2(b\Omega)$, we conclude that $z^n = q(z)$, i.e., that z^n is in the Szegő span. Since this argument works for any nonnegative integer n, we conclude that all complex polynomials are in the Szegő span.

To prove the converse, if the complex polynomials are in the Szegő span, then z and 1 are. The fact that 1 is in the span implies that the domain is an arc length quadrature domain. The function z is equal to $z/1$, which in turn is equal to a quotient of functions in the Szegő span. If we apply versions of identity (7.1) differentiated with respect to a to these functions, we see that the quotient is equal to the conjugate of a quotient of linear combinations of the Garabedian kernel and its derivatives in the second variable. Thus, $\bar{z}$ is equal to the boundary values of a meromorphic function, namely, the Schwarz function $S(z)$, and we conclude that Ω is an area quadrature domain, too. □

We close this chapter by remarking that it is possible to prove that any bounded domain with C^∞ smooth boundary can be approximated by a smooth arc length quadrature domain via a biholomorphic mapping that is as C^∞-close to the identity as desired. The arguments closely parallel those used in the last chapter, but are somewhat more complicated because of the square roots. A considerably more difficult result is to prove that any bounded domain with C^∞ smooth boundary can be approximated by a smooth *double* quadrature domain via a biholomorphic mapping that is as C^∞-close to the identity as desired. This result is an even better improvement upon the Riemann mapping theorem than our previous ones because double quadrature domains are even more like the unit disc. We refer the interested reader to the Bibliographic Notes for references to these results.