24

The Hilbert transform

Suppose that Ω is a bounded *simply connected* domain with C^∞ smooth boundary. Let $a \in \Omega$ be a fixed point. Given a *real valued* function u in $C^\infty(b\Omega)$, we may identify u as the boundary values of a real valued harmonic function U on Ω that is in $C^\infty(\overline{\Omega})$ (see Theorem 10.1). We will show that there is a real valued harmonic conjugate function V (meaning that $U + iV$ is holomorphic on Ω) such that V is also in $C^\infty(\overline{\Omega})$. We can make V uniquely determined by specifying that $V(a) = 0$. Under these conditions, let v denote the restriction to the boundary of V. The function v is called the *Hilbert transform* of u and we write $\mathcal{H}u = v$. In this chapter, we will prove that the Hilbert transform is a well defined linear operator and we will prove the classical facts that $\mathcal{H}$ maps $C^\infty(b\Omega)$ into itself and that $\mathcal{H}$ extends uniquely to be a bounded operator on $L^2(b\Omega)$.

Given a *real valued* function u in $C^\infty(b\Omega)$, we have expressed the harmonic extension U of u as a sum $h+\overline{H}$ where h and H are in $A^\infty(\Omega)$ and are given by $h = P(S_a u)/S_a$ and $H = P(L_a \bar{u})/L_a$ (Theorem 10.1). Since u is real valued, the maximum principle shows that its harmonic extension U is also real valued, and it follows that the imaginary parts of h and H must be equal. Hence, h and H differ by a real constant on Ω. In fact, since $H(a) = 0$, we deduce that $h(z) = H(z) + h(a)$ for all $z \in \Omega$, and therefore, that the harmonic extension of u is given by

$$U(z) = h(a) + 2\text{Re } H(z).$$

We have expressed the harmonic extension of u as the real part of the holomorphic function $h(a)+2H(z)$, and we may write $U+iV = h(a)+2H$. The real valued function V in this formula is a harmonic conjugate function for U and it is uniquely determined by the condition that $V(a) = 2H(a) = 0$. The *Hilbert transform* $\mathcal{H}u$ of u is defined to be the function v on $b\Omega$ given by the boundary values of V. We have shown that $\mathcal{H}u = 2\text{Im } H$.

Since Im h = Im H, we may summarize our work above in the following theorem.

Theorem 24.1. *The Hilbert transform on a bounded simply connected* *domain Ω with C^∞ smooth boundary is given by*

$$\mathcal{H}u = 2Im\left(\frac{P(L_a\,u)}{L_a}\right),$$

or

$$\mathcal{H}u = 2Im\left(\frac{P(S_a\,u)}{S_a}\right) \tag{24.1}$$

where P denotes the Szegő projection associated to Ω. It follows that the Hilbert transform is a linear operator mapping $C^\infty(b\Omega)$ into itself that extends uniquely to be a bounded operator from $L^2(b\Omega)$ to itself.

Indeed, since S_a is a nonvanishing function in $C^\infty(b\Omega)$ and since P is a bounded operator on $L^2(b\Omega)$, the fact that the Hilbert transform extends to be a bounded operator on $L^2(b\Omega)$ can be read off from formula (24.1).

Let f denote the Riemann mapping function that maps Ω one-to-one onto the unit disc with $f(a) = 0$ and $f'(a) > 0$. It was shown in Chapter 12 that the Szegő kernel transforms under the map f via

$$S(z,a) = \sqrt{f'(z)}S_U(f(z), f(a))\overline{\sqrt{f'(a)}}, \tag{24.2}$$

where $S_U(z,w)$ denotes the Szegő kernel of the unit disc. Now, $S_U(z,0) \equiv (2\pi)^{-1}$. Hence, it follows from (24.2) that

$$S(z,a) = (2\pi)^{-1}\sqrt{f'(z)}\;\overline{\sqrt{f'(a)}}. \tag{24.3}$$

When this formula is plugged into (24.1), we obtain the next theorem.

Theorem 24.2. *The Hilbert transform can be written*

$$\mathcal{H}u = 2Im\left(\frac{P\left(u\sqrt{f'}\right)}{\sqrt{f'}}\right)$$

where f is the Riemann mapping function that maps Ω one-to-one onto the unit disc with $f(a) = 0$ and $f'(a) > 0$.

Formula (24.1) can be used to estimate the constant C in the L^2 estimate $\|\mathcal{H}u\| \le C\|u\|$. Indeed, we have

$$\|\mathcal{H}u\| \le 2\|S_a^{-1}P(S_a u)\| \le 2\left(\operatorname*{Max}_{w\in b\Omega}|S_a(w)^{-1}|\right)\|P(S_a u)\|,$$

and

$$\|P(S_a u)\| \le \|S_a u\| \le \left(\operatorname*{Max}_{w\in b\Omega}|S_a(w)|\right)\|u\|.$$

Let

$$M = \operatorname*{Max}_{w \in b\Omega} |S(w,a)|, \text{ and}$$
$$m = \min_{w \in b\Omega} |S(w,a)|.$$

We have proved that the smallest possible constant C in the L^2 estimate $\|\mathcal{H}u\| \leq C\|u\|$ for the Hilbert transform satisfies

$$C \leq 2\frac{M}{m}.$$

The fraction M/m can also be described in terms of the Riemann mapping function f that maps Ω one-to-one onto the unit disc with $f(a) = 0$ and $f'(a) > 0$. Let

$$\Lambda = \operatorname*{Max}_{w \in b\Omega} |f'(w)|, \text{ and}$$
$$\lambda = \min_{w \in b\Omega} |f'(w)|.$$

Then, using (24.3), we see that $M/m = \Lambda^{1/2}/\lambda^{1/2}$.

We mention one last formula. The Szegő projection transforms under the Riemann map f according to the identity

$$P\left(\sqrt{f'}(\varphi \circ f)\right) = \sqrt{f'}\left((P_U\varphi) \circ f\right),$$

where P_U denotes the Szegő projection on the unit disc U and $\varphi \in L^2(bU)$. Apply this identity to the function $\varphi = u \circ f^{-1}$ and plug the result into the formula of Theorem 24.2 to obtain

$$\mathcal{H}u = 2\text{Im}\ \left(P_U(u \circ f^{-1})\right) \circ f.$$

Actually, this last formula is not hard to derive from first principles and it is possible to make this result the starting point of the theory as an alternative to basing the reasoning on Theorem 4.3 and identity (7.1).

25

The Bergman kernel and the Szegő kernel

The reader should suspect that, because boundary integrals can readily be turned into solid integrals by means of the Green's formula, the Bergman kernel and the Szegő kernel of a domain should be closely related. In this chapter, we show that they are very closely related indeed.

Theorem 25.1. *If Ω is a bounded simply connected domain with C^∞ smooth boundary, then the Bergman kernel $K(z,a)$ associated to Ω is related to the Szegő kernel via the identity*

$$K(z,a) = 4\pi S(z,a)^2.$$

To prove this identity, we use the relationships that exist between the derivative of a Riemann map and the two kernels. Let f be a biholomorphic map of Ω onto the unit disc such that $f(a) = 0$ and $f'(a) > 0$. It was shown in Chapter 15 that $f'(z) = C\,K(z,a)$ where $C = \sqrt{\pi/K(a,a)}$. Theorem 12.3 states that $f'(z) = cS(z,a)^2$ where $c = 2\pi/S(a,a)$. Hence it follows that $K(z,a) = (c/C)S(z,a)^2$. By plugging in $z = a$ into this last formula and by using the expressions for c and C, it can be deduced that $c/C = 4\pi$.

If Ω is multiply connected, then the relationship between the kernels is not as direct. In the next theorem, the functions $F_j'(z)$ denote the derivatives of the classical harmonic measure functions that were introduced in Chapter 19.

Theorem 25.2. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Then the Bergman kernel and Szegő kernel are related via*

$$K(z,a) = 4\pi S(z,a)^2 + \sum_{j=1}^{n-1} \lambda_j F_j'(z),$$

where the coefficients λ_j are constants in z which depend on a.

Proof. Define $G(z) = K(z,a) - 4\pi S(z,a)^2$. To prove the theorem, we will show that GT is orthogonal in $L^2(b\Omega)$ to $H^2(b\Omega)$ and to the space of conjugates of functions in $H^2(b\Omega)$. Then Theorem 19.1 implies that $G = \sum \lambda_j F_j'$ and this is what we want to see. Since $K(z,a)$ and $S(z,a)$ are in $A^\infty(\Omega)$ as functions of z for fixed $a \in \Omega$, Theorem 4.3 yields immediately that GT is orthogonal to the space of conjugates of functions in $H^2(b\Omega)$. To see that GT is orthogonal to $H^2(b\Omega)$, let $h \in A^\infty(\Omega)$ and compute

$$\begin{aligned}\langle K_a T, h\rangle_{b\Omega} &= \int_{b\Omega} K_a T\,\overline{h}\,ds = \int_{b\Omega} K_a\,\overline{h}\,dz \\ &= \iint_\Omega K_a\,\overline{h'}d\bar{z}\wedge dz = 2i\,\langle K_a, h'\rangle_\Omega = 2i\,\overline{h'(a)}\end{aligned}$$

(by the reproducing property of the Bergman kernel). Next, observe that, by identity (7.1),

$$(S_a)^2 T = -(\overline{L_a})^2\overline{T}.$$

By integrating this identity around the boundary with respect to ds, we see that the residue of $(L_a)^2$ at $z = a$ is zero by applying the residue theorem on the right hand side and Cauchy's theorem on the left. Now we may compute

$$\begin{aligned}\langle (S_a)^2 T, h\rangle_{b\Omega} &= \int_{b\Omega} (S_a)^2 T\,\overline{h}\,ds = -\int_{b\Omega} \overline{(L_a)^2 T\,h}\,ds \\ &= -\int_{b\Omega} \overline{(L_a)^2\,h}\,d\bar{z} = -(-2\pi i)\frac{1}{(2\pi)^2}\overline{h'(a)}\end{aligned}$$

because $(2\pi L_a)^2 = (z-a)^{-2} + H$ where $H \in A^\infty(\Omega)$. Hence, we have shown that $\int_{b\Omega} GT\,\overline{h}\,ds = 0$. Since $A^\infty(\Omega)$ is dense in $H^2(b\Omega)$, it follows that GT is orthogonal to $H^2(b\Omega)$, and the proof is complete. □

The numbers λ_j in Theorem 25.2 are functions of a. In fact, it is not hard to show that $\lambda_j(a)$ is an antiholomorphic function of a given by

$$\lambda_j(a) = \sum_{k=1}^{n-1} C_{jk}\overline{F_k'(a)}$$

for some constants C_{jk}. To see this, we integrate the formula in Theorem 25.2 around one of the $n-1$ inner boundary curves γ_k to obtain

$$\int_{\gamma_k} \left(K(z,a) - 4\pi S(z,a)^2\right)\,dz = \sum_{j=1}^{n-1}\lambda_j \int_{\gamma_k} F_j'(z)\,dz = \sum_{j=1}^{n-1} A_{kj}\lambda_j$$

where $[A_{kj}]$ denotes the nonsingular matrix of periods that we discussed in Chapter 19. Let γ_k^ϵ represent a curve that is homotopic to γ_k, but that is inside Ω (such as the curve traced out by a point at a distance of

ϵ along the inward pointing unit normal vector to γ_k for small ϵ.) We may now write

$$\int_{\gamma_k^\epsilon} \left(K(z,a) - 4\pi S(z,a)^2\right)\, dz = \sum_{j=1}^{n-1} A_{kj}\lambda_j(a).$$

This shows that the $\lambda_j(a)$ are antiholomorphic functions of a because the kernels are antiholomorphic in a. Now consider the idea of approximating the integral in the identity above by a finite Riemann sum. Since $K(z,a) - 4\pi S(z,a)^2$ is the conjugate of $K(a,z) - 4\pi S(a,z)^2$, it follows from Theorem 25.2 that $K(z,a) - 4\pi S(z,a)^2$ is in the linear span of $\{\overline{F_k'(a)}\}_{k=1}^{n-1}$ for each fixed z. Hence, as a function of a, the Riemann sum represents a function in the linear span of the $\overline{F_k'(a)}$. A simple limiting argument now shows that a limit of such Riemann sums must also lie in this span and the proof is complete.

Therefore, the Bergman and Szegő kernels are actually related via

$$K(z,a) = 4\pi S(z,a)^2 + \sum_{j,k=1}^{n-1} C_{jk} F_j'(z)\overline{F_k'(a)} \tag{25.1}$$

for some constants C_{jk}.

We may use the results of Chapter 19 to write the formula relating the Szegő kernel to the Bergman kernel in a different form. Let us use the notation $S'(z,w)$ to denote the function $\frac{\partial}{\partial z}S(z,w)$, i.e., the prime denotes differentiation in the holomorphic variable.

Theorem 25.3. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. For a point $a \in \Omega$, suppose that the zeroes of the Szegő kernel $S(z,a)$ are given as the set $\{a_j\}_{j=1}^{n-1}$ of $n-1$ distinct points in Ω. The Bergman kernel associated to Ω is related to the Szegő kernel via the identity*

$$K(z,a) = 4\pi S(z,a)^2 + 2\pi \sum_{j=1}^{n-1} \frac{K(a_j,a)}{S'(a_j,a)} L(z,a_j)S(z,a)$$

Proof. Since, by Theorem 19.1, the linear span of the functions F_j' is the same as the linear span of the functions $L(z,a_j)S(z,a)$, it is clear from Theorem 25.2 that there are constants c_j such that

$$K(z,a) = 4\pi S(z,a)^2 + 2\pi \sum_{j=1}^{n-1} c_j L(z,a_j)S(z,a).$$

The values of the constants c_j are easily determined because the functions $G_j(z) = L(z, a_j)S(z, a)$ are such that

$$G_j(a_k) = 0 \qquad \text{if } k \neq j, \text{ and}$$
$$G_j(a_j) = \frac{S'(a_j, a)}{2\pi}$$

since $S(a_k, a) = 0$ for each k and $L(z, a_j)$ has a single simple pole at $z = a_j$ with residue $1/(2\pi)$. The proof is finished. □

Another way to relate the Bergman kernel to the Szegő kernel is by means of the connection of both of these objects to the Dirichlet problem. We will now show that the Bergman kernel associated to Ω is directly related to the Poisson extension of the boundary values of the function $1/(z - a)$ to Ω.

Theorem 25.4. *Let Ω denote a bounded domain with C^∞ smooth boundary. Let $a \in \Omega$ and let u denote the Poisson extension to Ω of $\varphi(z) = (2\pi i)^{-1}/(z - a)$. The Bergman kernel function $K(z, a)$ is given by*

$$K(z, a) = -2i\frac{\partial \bar{u}}{\partial z}.$$

Proof. If $h \in A^\infty(\Omega)$, then

$$h(a) = \int_{b\Omega} h\,\varphi\;dz = \iint_\Omega h\;\frac{\partial u}{\partial \bar{z}}\;d\bar{z} \wedge dz$$
$$= \iint_\Omega h(z)\;\overline{G(z)}\;dx \wedge dy$$

where $G(z)$ is the holomorphic function on Ω given by $-2i(\partial\bar{u}/\partial z)$. Thus, the inner product of a function $h \in A^\infty(\Omega)$ with G is equal to the value of h at a. Since $A^\infty(\Omega)$ is dense in the Bergman space, this holds true for all h in the Bergman space. This reproducing property characterizes the Bergman kernel, and therefore $K(z, a) = G(z)$ and the theorem is proved. □

The formula in Theorem 25.4 is most interesting on a multiply connected domain. On a simply connected domain, the Bergman kernel $K(z, a)$ is a constant times the derivative of the Riemann mapping function mapping a to the origin given by S_a/L_a. From this, it is not hard to deduce that

$$K(z, a) = 2S(a, a)\frac{\partial}{\partial z}\left(\frac{S(z, a)}{L(z, a)}\right).$$

We proved in Chapter 16 that the Bergman projection and kernel transform under proper holomorphic mappings. We close this chapter by showing that the Szegő kernel transforms under *certain* proper holomorphic maps. In order to prove this result, we will need a fact about proper holomorphic maps that follows from the material in Chapter 16.

Suppose g is a proper holomorphic mapping of a bounded multiply connected domain Ω_1 with C^∞ smooth boundary onto a bounded *simply connected* domain Ω_2 with C^∞ smooth boundary. Suppose that the multiplicity of g is m. Let V_2 denote the discrete subset of Ω_2 that is the image of the branch locus of g. Let G_k, $k = 1, \ldots, m$, denote the local inverses to g. We proved in Chapter 16 that the G_k are defined on $\Omega_2 - V_2$, that they extend smoothly to the boundary of Ω_2, and that they map the boundary of Ω_2 into the boundary of Ω_1.

If ω_j is the harmonic measure function equal to one on the j-th boundary curve of Ω_1 and equal to zero on the others, then

$$\sum_{k=1}^{m} \omega_j(G_k(w)) \tag{25.2}$$

is equal to a constant (which is a positive integer) for $w \in \Omega_2 - V_2$. To prove this fact, notice that the sum defines a positive harmonic function on $\Omega_2 - V_2$. Since this harmonic function is bounded above by m and below by zero, and since the set V_2 is discrete, the possible singularities at points in V_2 are removable. Thus, we may think of the sum as defining a harmonic function u on all of Ω_2. Since the functions G_k extend smoothly to $b\Omega_2$ and since they map $b\Omega_2$ into $b\Omega_1$, it follows that u extends smoothly to $\overline{\Omega}_2$ and that it is positive integer valued on $b\Omega_2$. The boundary of Ω_2 is connected since Ω_2 is simply connected. A continuous integer valued function on a connected set must be constant. Hence, u is constant on $b\Omega_2$ and the maximum principle shows that u is constant on $\overline{\Omega}_2$.

By differentiating equation (25.2) with respect to z and using the complex chain rule, we see that

$$0 \equiv \sum_{k=1}^{m} F_j'(G_k(w))G_k'(w). \tag{25.3}$$

Recall that the operator Λ_2 that we used in Chapter 16 was defined as $\Lambda_2 u = \sum_{k=1}^{m} G_k'(u \circ G_k)$, and thus, identity (25.3) can be expressed very simply in terms of the operator Λ_2 as $\Lambda_2 F_j' = 0$. Now, apply Λ_2 in the z variable to the identity in Theorem 25.2 to obtain

$$\Lambda_2 K_1(\cdot, a) = 4\pi \Lambda_2 S_1(\cdot, a)^2.$$

Note that the F_j' terms drop out because $\Lambda_2 F_j' = 0$. The transformation formula for the Bergman kernels yields that $\Lambda_2 K_1(\cdot, a) =$

$K_2(\cdot, g(a))\overline{g'(a)}$, which in a simply connected domain, is equal to $4\pi S_2(\cdot, g(a))^2\overline{g'(a)}$. Hence, we have proved that, under these conditions the Szegő kernel transforms under g via

$$g'(z)S_2(g(z), w)^2 = \sum_{k=1}^{m} S_1(z, G_k(w))^2\overline{G_k'(w)}.$$

It is worth pointing out that an Ahlfors map associated to a bounded multiply connected domain with C^∞ smooth boundary satisfies the conditions under which this transformation formula has been proved to be valid.

26

Pseudo-local property of the Cauchy transform and consequences

Let Ω denote a bounded domain with C^∞ smooth boundary. The Cauchy transform associated to Ω is an example of an operator that is not *local.* A local operator $\mathcal{Q}$ on $L^2(b\Omega)$ would have the property that, given a function $u \in L^2(b\Omega)$ that vanishes on an open arc A in the boundary, then $\mathcal{Q}u$ also vanishes on this arc. It is easy to see that the Cauchy transform does not satisfy this property because the transform of a function vanishing on an arc extends holomorphically past that arc. However, the Cauchy transform is an example of a *pseudo-local* operator. This means that, given an open connected arc contained in the boundary of a bounded domain Ω with C^∞ smooth boundary, if a function in $L^2(b\Omega)$ is C^∞ smooth on this arc, then so is its Cauchy transform. In this chapter, we will study this property in more detail and deduce some of its consequences.

If A is an open connected arc in the boundary of Ω, we will let $\|u\|_s^A$ denote the C^s norm of a function u on A. To be precise, fix a C^∞ parameterization $z(t)$ of the arc A such that $z(t)$ traces out the arc as t ranges from a to b. We define $\|u\|_s^A$ to be the supremum of $|\frac{\partial^k}{\partial t^k}u(z(t))|$ over $a < t < b$ and $0 \leq k \leq s$. We will continue to use the unadorned symbol $\|u\|$ to denote the $L^2(b\Omega)$ norm of u.

Let γ be an open connected arc contained in the boundary of Ω and let Γ be another such arc that compactly contains γ. The Cauchy transform satisfies the following property known as a *pseudo-local estimate.*

Theorem 26.1. *Given a positive integer s, there is a positive integer $n = n(s)$ and a constant $K = K(s)$ such that*

$$\|\mathcal{C}u\|_s^\gamma \leq K\left(\|u\|_n^\Gamma + \|u\|\right)$$

for all $u \in L^2(b\Omega)$.

It should be remarked that part of the conclusion of this theorem is that if the norms on the right hand side of the inequality make sense, then the norm on the left hand side makes sense too.

It is possible to prove an improved version of Theorem 26.1 in which

n is taken to be equal to $s+1$ but we will not prove this version. It will suffice for us to know only that the estimate holds for *some* value of n.

Proof. Theorem 26.1 is a straightforward consequence of Theorem 9.2. To see this, let χ be a function in $C^\infty(b\Omega)$ that is equal to one on a neighborhood of the closure of γ in $b\Omega$ and equal to zero on $b\Omega - \Gamma$. Now, we may write $\mathcal{C}u = \mathcal{C}(\chi u) + \mathcal{C}((1-\chi)u)$. Because $(1-\chi)u$ is supported away from γ, it is clear that we may differentiate under the integral sign defining the Cauchy transform and apply Hölder's inequality to see that

$$\|\mathcal{C}((1-\chi)u)\|_s^\gamma \leq (\text{constant}) \left(\int_{b\Omega-\gamma} |u|^2 ds \right)^{1/2} \leq (\text{constant})\|u\|.$$

To analyze the other part of $\mathcal{C}u$, we use Theorem 9.2 to obtain

$$\|\mathcal{C}(\chi u)\|_s^\gamma \leq K\|\chi u\|_n^{b\Omega} \leq (\text{constant})\|u\|_n^\Gamma.$$

Combining these two inequalities now yields the theorem. □

Recall that, as a consequence of the Kerzman-Stein identity, the Szegő projection is related to the Cauchy transform via identity (4.3),

$$P = \mathcal{C} - \mathcal{A}(I-P).$$

Because the Kerzman-Stein operator $\mathcal{A}$ is given by integration against a kernel in $C^\infty(b\Omega \times b\Omega)$, it follows that

$$\|\mathcal{A}(I-P)u\|_s^\gamma \leq C\|(I-P)u\| \leq C\|u\|.$$

Thus, the pseudo-local estimate for the Cauchy transform implies the same kind of estimate for the Szegő projection.

Theorem 26.2. *Given a positive integer s, there is a positive integer $n = n(s)$ and a constant $K = K(s)$ such that*

$$\|Pu\|_s^\gamma \leq K\left(\|u\|_n^\Gamma + \|u\|\right)$$

for all $u \in L^2(b\Omega)$.

As was the case for Theorem 26.1, Theorem 26.2 can be proved using $n = s+1$, but we will not need this refinement.

One of the most interesting consequences of this pseudo-local estimate is the following result.

Theorem 26.3. *The Szegő kernel function is in $C^\infty((\overline{\Omega}\times\overline{\Omega})-\Delta)$ where $\Delta = \{(z,z) : z \in b\Omega\}$ denotes the boundary diagonal set.*

In view of identity (25.1), it follows from Theorem 26.3 and the fact that the functions F_j' are all in $A^\infty(\Omega)$ that Bergman kernel $K(z,w)$ is also in $C^\infty((\overline{\Omega}\times\overline{\Omega})-\Delta)$.

Proof. Let z_0 and w_0 be distinct points in the boundary of Ω and let $\epsilon>0$ be small enough that the closures of the balls of radius ϵ about z_0 and w_0 do not intersect. As in Chapter 7, let $C_w(\zeta)$ denote the complex conjugate of

$$\frac{1}{2\pi i}\frac{T(\zeta)}{\zeta-w}.$$

For z and w in Ω, we know that

$$S(z,w)=(PC_w)(z)=(\mathcal{C}C_w)(z)-(P\mathcal{A}C_w)(z)$$

where the last equality follows from the Kerzman-Stein identity $P(I+\mathcal{A})=\mathcal{C}$. Let us define $\mathcal{H}_1(z,w)=(\mathcal{C}C_w)(z)$ and $\mathcal{H}_2(z,w)=(P\mathcal{A}C_w)(z)$. The term $\mathcal{H}_1(z,w)$ is the interesting part of $S(z,w)$; the term $\mathcal{H}_2(z,w)$ turns out to be very well behaved.

To analyze $\mathcal{H}_2(z,w)$, notice that, for $\zeta\in b\Omega$,

$$(\mathcal{A}C_w)(\zeta)=\int_{\xi\in b\Omega}A(\zeta,\xi)C_w(\xi)\,ds,$$

and, because $C_w(\xi)$ is the conjugate of the kernel function of the Cauchy transform, this is equal to the complex conjugate of the Cauchy transform of the function $\psi_\zeta(\xi)=\overline{A(\zeta,\xi)}$ evaluated at w. Now, because these functions and their derivatives in ξ are bounded on $b\Omega$ as ζ ranges over the boundary, and because the Cauchy transform satisfies the estimate in Theorem 9.2, it follows that $(\mathcal{A}C_w)(\zeta)$ and all its derivatives in w are bounded on Ω as ζ ranges over $b\Omega$. We will now repeat this argument for derivatives of $(\mathcal{A}C_w)(\zeta)$ with respect to ζ. Let $\zeta(t)$ represent a parameterization of the boundary and, when w is taken to be a fixed point in Ω, consider derivatives of $(\mathcal{A}C_w)(\zeta(t))$ with respect to t. The k-th derivative is given by

$$\int_{\xi\in b\Omega}\frac{\partial^k}{\partial t^k}A(\zeta(t),\xi)C_w(\xi)\,ds,$$

which is equal to the complex conjugate of the Cauchy transform of the function $\psi_\zeta^k(\xi)$ defined to be the complex conjugate of $(\partial^k/\partial t^k)A(\zeta(t),\xi)$. Since this function and its derivatives with respect to ξ are bounded on $b\Omega$, we may repeat the reasoning above to see that the function $(\partial^k/\partial t^k)(\mathcal{A}C_w)(\zeta(t))$ and all its derivatives in w are bounded on Ω as ζ ranges over $b\Omega$. We have shown that $(\mathcal{A}C_w)(\zeta)$ is in $C^\infty(b\Omega\times\overline{\Omega})$ as a function of (ζ, w). If we now apply the Szegő projection in ζ, we can use the uniform estimate of Theorem 9.2 for P to deduce that $\mathcal{H}_2(z,w)$ is in $C^\infty(\overline{\Omega} \times \overline{\Omega})$.

We now assume that $z \in D_\epsilon(z_0) \cap \Omega$ and $w \in D_\epsilon(w_0) \cap \Omega$. To analyze $\mathcal{H}_1(z,w)$, we let χ be a function in $C^\infty(b\Omega)$ such that $\chi = 1$ in $D_\epsilon(z_0) \cap b\Omega$ and $\chi = 0$ in $D_\epsilon(w_0) \cap b\Omega$. We now split $\mathcal{H}_1(z,w)$ into two pieces via

$$\mathcal{H}_1(z,w) = (\mathcal{C}(\chi C_w))(z) + (\mathcal{C}((1-\chi)C_w))(z).$$

Consider the first term in this sum. Notice that $(\chi C_w)(\zeta)$ and its derivatives in ζ are bounded on $b\Omega$ as w ranges over $D_\epsilon(w_0) \cap \overline{\Omega}$. Furthermore, derivatives of $(\chi C_w)(\zeta)$ with respect to w give rise to functions of ζ with the same property. Thus, it follows that the first term in the sum for $\mathcal{H}_1$ is in $C^\infty((D_\epsilon(z_0) \cap \overline{\Omega}) \times (D_\epsilon(w_0) \cap \overline{\Omega}))$.

The second term can also be seen to belong to this class by observing that

$$(\mathcal{C}((1-\chi)C_w))(z) = \overline{(\mathcal{C}((1-\bar{\chi})C_z))(w)},$$

and the same reasoning can be applied to this term. □

In the course of the proof of Theorem 26.3, we proved that

$$S(z,w) = \frac{1}{4\pi^2} \int_{\zeta \in b\Omega} \frac{1}{(\zeta - z)(\bar{\zeta} - \bar{w})}\, ds + H(z,w)$$

where $H(z,w)$ is in $C^\infty(\overline{\Omega} \times \overline{\Omega})$ and where the integral is equal to $\mathcal{H}_1(z,w) = (\mathcal{C}C_w)(z)$. By letting $z = a$ and $w = a$ in this formula, it becomes clear that $S(a,a)$ tends to $+\infty$ as a tends to a point in the boundary at the same rate as

$$\frac{1}{4\pi^2} \int_{\zeta \in b\Omega} \frac{1}{|\zeta - a|^2}\, ds.$$

This last integral can easily be seen to blow up like a constant times the inverse of the distance from a to the boundary.

We proved in Chapter 10 that the Poisson kernel on a bounded simply connected domain with C^∞ smooth boundary is given by

$$p(z,w) = \frac{S(z,w)S(w,a)}{S(z,a)} + \frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}} = \frac{|S(z,w)|^2}{S(z,z)}.$$

We can now assert that the Poisson kernel has *all* the familiar properties that we admire about the Poisson kernel on the unit disc. We can now add to the properties proved in Chapter 10 that $p(z,w)$ is in $C^\infty((\overline{\Omega} \times b\Omega) - \Delta)$ where $\Delta = \{(z,z) : z \in b\Omega\}$. Also, for a fixed point $w_0 \in b\Omega$ and $\delta > 0$, $p(z,w)$ tends to zero uniformly in w on the set $b\Omega - D_\delta(w_0)$ as z

tends to the boundary while staying in the set $\Omega \cap D_{\delta/2}(w_0)$. When these two properties are added to the ones proved in Chapter 10, we know all the properties of the Poisson kernel needed to prove Schwarz's theorem about the solution of the Dirichlet problem with continuous boundary data.

We now turn to the study of the behavior of the Garabedian kernel $L(z, a)$ when z and a are both close to the boundary.

Theorem 26.4. *If Ω is a bounded domain with C^∞ smooth boundary, then the function $\ell(z, w)$ defined via*

$$L(z, w) = \frac{1}{2\pi(z-w)} + \ell(z, w)$$

is a function on $\Omega \times \Omega$ that is holomorphic in z and w and that extends to be in $C^\infty(\overline{\Omega} \times \overline{\Omega})$.

Proof. We claim that, as a function of z, $\ell(z, a)$ is the Szegő projection of the function G_a defined to be $(2\pi)^{-1}(z-a)^{-1}$. To see this, note that $\ell(z, a)$ is holomorphic in z on Ω. Hence

$$\ell(z, a) = P(\ell(\cdot, a)) = PL_a - PG_a = -PG_a$$

because $L_a = i\,\overline{S_a T}$ is orthogonal to holomorphic functions.

Now the Kerzman-Stein identity $P(I + \mathcal{A}) = \mathcal{C}$ allows us to write

$$PG_a = \mathcal{C}G_a - P\mathcal{A}G_a.$$

A simple calculation using the residue theorem reveals that $(\mathcal{C}G_a)(z)$ is zero for all $z \in \Omega$ and $a \in \Omega$. Hence, the proof will be finished if we prove that $(\mathcal{A}G_a)$ is in $C^\infty(\overline{\Omega} \times \overline{\Omega})$. But

$$(\mathcal{A}G_a)(z) = \frac{1}{2\pi} \int_{\xi \in b\Omega} A(z, \xi) \frac{1}{\xi - a}\, ds,$$

and this integral represents the Cauchy transform of the function $\psi_z(\xi) = iA(z, \xi)\overline{T(\xi)}$ evaluated at a. Hence, we may reason exactly as we did in the proof of Theorem 26.3 to see that all the mixed derivatives of $(\mathcal{A}G_a)(z)$ in a and z are bounded on $b\Omega \times \Omega$. This completes the proof. □

There are even stronger theorems about the boundary behavior of the Szegő and Garabedian kernels in domains with real analytic boundaries.

Theorem 26.5. *On a bounded domain with real analytic boundary, the Szegő kernel $S(z, w)$ extends to be defined on a neighborhood of*

$(\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in b\Omega\}$ as a function that is holomorphic in z and antiholomorphic in w, and the Garabedian kernel is given by

$$L(z, w) = \frac{1}{2\pi(z - w)} + \ell(z, w),$$

where $\ell(z, w)$ extends to be holomorphic in z and w on a neighborhood of $(\overline{\Omega} \times \overline{\Omega})$.

Note that, because the functions F_j' all extend holomorphically to a neighborhood of $\overline{\Omega}$, it follows from Theorem 26.5 and identity (25.1), that the Bergman kernel $K(z, w)$ enjoys the same extension properties as the Szegő kernel.

Proof. Suppose that Ω is a bounded domain with real analytic boundary, and let γ denote one of the boundary curves of Ω. We will need to use a reflection function for this real analytic curve as described in Chapter 11. Let $R(z)$ denote an antiholomorphic reflection function for γ. For example, $1/\bar{z}$ is an antiholomorphic reflection function for the unit circle. Recall that such a function is defined and antiholomorphic in a neighborhood of γ, fixes γ, and is locally diffeomorphic near γ. Near γ, $R(z)$ maps the outside of Ω to the inside, and the inside to the outside, and $R(R(z)) = z$. We may define a reflection like this for each of the boundary curves of Ω; we will use the same symbol R to denote each of them. In this way, we may view R as an antiholomorphic function defined on a small neighborhood of $b\Omega$.

Fix a point a in Ω and let w be another point in Ω that we will allow to vary. By (7.1), we may write $S(a, z) = -iL(z, a)T(z)$ and $S(w, z) = -iL(z, w)T(z)$ when $z \in b\Omega$. After dividing the second equation by the first, and using the fact that $R(z) = z$ on the boundary, we may write

$$\frac{S(w, R(z))}{S(a, R(z))} = \frac{L(z, w)}{L(z, a)} \qquad \text{for } z \in b\Omega. \tag{26.1}$$

The function on the left hand side of this equality is defined and holomorphic for z near γ on the outside of Ω. We know by Theorem 11.2 that $L(z, w)$ extends holomorphically past the boundary as a function of z for each fixed $w \in \Omega$. Hence the function on the left hand side agrees with the holomorphic extension of $L(z, w)/L(z, a)$ outside of Ω. A particularly interesting consequence of this formula is that $L(z, w)/L(z, a)$ is seen to extend holomorphically to a neighborhood of γ *that is independent of* w. Now consider what (26.1) implies as w is allowed to tend to a boundary point $w_0 \in b\Omega$. Let w_k be a sequence in Ω tending to w_0. We deduce that there is a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ such that, as a function of z, each $L(z, w_k)$ extends holomorphically to $\mathcal{O} - \{w_k\}$, and as w_k tends to w_0, these functions converge uniformly on compact subsets of $\mathcal{O} - \{w_0\}$ to a function $L_0(z)$ that is holomorphic on $\mathcal{O} - \{w_0\}$. By writing contour integrals about a small fixed circle around w_0 that give the coefficients of the Laurent expansion of $L(z, w_k)$ about the point w_k, and by taking uniform limits under integral signs, it is seen that $L_0(z)$ has a simple pole at w_0 with residue $1/(2\pi)$. Since $L_0(z)$ agrees with $L(z, w_0)$ inside Ω, we have produced a meromorphic extension of $L(z, w_0)$. We now abandon our $L_0(z)$ notation and allow $L(z, w_0)$ to denote the meromorphic function defined on $\mathcal{O}$. Note that since $L(z, w) = -L(w, z)$, we may deduce the same extension property in the w variable.

We have now defined $L(z, w)$ for (z, w) in $(\mathcal{O} \times \overline{\Omega}) \cup (\overline{\Omega} \times \mathcal{O})$. We will complete the proof by defining $L(z, w)$ on the rest of $\mathcal{O} \times \mathcal{O}$. It follows from Theorems 26.3 and 26.4 that identity (7.1) holds even when both variables are on the boundary, that is

$$S(w, z) = -iL(z, w)T(z) \qquad \text{for } z, w \in b\Omega,\ z \neq w.$$

Hence, if $z, w \in b\Omega$ and $z \neq w$, we may use (7.1) in both variables to write

$$-iT(w)L(z, w)T(z) = T(w)S(w, z) = \overline{-iL(w, z)}.$$

Hence, if $z, w \in b\Omega$, and $z \neq w$, we have

$$T(z)L(z, w)T(w) = \overline{L(z, w)}. \tag{26.2}$$

Fix a point a in Ω and notice that $T(\zeta) = i\,\overline{S_a(\zeta)}/L_a(\zeta)$. Plugging this into the last identity yields

$$-\frac{L(z, w)}{L_a(z)L_a(w)} = \overline{\frac{L(z, w)}{S_a(z)S_a(w)}}$$

when $z, w \in b\Omega$, $z \neq w$. Using the fact that $\zeta = R(\zeta)$ when $\zeta \in b\Omega$, we may write

$$-\frac{L(z, w)}{L_a(z)L_a(w)} = \overline{\frac{L(R(z), R(w))}{S_a(R(z))S_a(R(w))}}.$$

This identity holds when z and w are on the boundary. The function on the left extends to be holomorphic for z, w inside Ω, $z \neq w$, the function on the right extends to be holomorphic for z, w outside Ω, $z \neq w$. We may reason as we did above to see that the function on the right hand side of this identity defines the meromorphic extension of the function on the left to z and w that are outside of Ω. It also follows that the extension is holomorphic in both variables z and w when $z \neq w$ and the singularity at $z = w$ is a simple pole with residue $1/(2\pi)$. We have now produced a meromorphic extension of $L(z, w)$ to $\mathcal{O} \times \mathcal{O}$. When the principal part $(2\pi)^{-1}(z-w)^{-1}$ is subtracted off, we see that $\ell(z,w)$ extends holomorphically to $\mathcal{O}\times\mathcal{O}$. Finally the extendibility of the Szegő kernel follows from that of $L(z,w)$ via identity (7.1) and the fact that holomorphic functions with real analytic boundary values must extend past the boundary. □

The function $\ell(z,w)$ pops up in many places in the study of conformal mapping. We close this chapter by deriving Burbea's formula [Bu1] which relates $\ell(z,w)$ to the Kerzman-Stein kernel. Identity (26.2) allows us to write

$$L(z,w)T(z) - \overline{L(z,w)T(w)} = 0$$

for $z,w \in b\Omega$, $z \neq w$. Since $L(z,w) = \ell(z,w) + (2\pi)^{-1}(z-w)^{-1}$, this means that

$$\ell(z,w)T(z) - \overline{\ell(z,w)T(w)} = -\frac{T(z)}{(2\pi)(z-w)} + \frac{\overline{T(w)}}{(2\pi)(\bar{z}-\bar{w})}.$$

The right hand side of this last formula is $-iA(w,z)$, and we have deduced Burbea's formula,

$$\ell(z,w)T(z) - \overline{\ell(z,w)T(w)} = -iA(w,z).$$

27

Zeroes of the Szegő kernel

In this chapter, we determine the behavior of the zeroes of the function $S_a(z)$ as a tends to a point in the boundary. We suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. We know that, for $a \in \Omega$ the function

$$f_{(a)}(z) = \frac{S(z,a)}{L(z,a)} \tag{27.1}$$

is the Ahlfors mapping associated to Ω, which is a branched n-to-one covering map of Ω onto the unit disc (see Chapter 13). Notice that $f_{(a)}(a) = 0$ because of the pole of $L(z,a)$ at $z = a$ and that $f'_{(a)}(a)$ is equal to $2\pi S(a,a)$. The n-to-one map $f_{(a)}$ must have $n-1$ other zeroes besides the one at a; these zeroes coincide with the zeroes of $S(z,a)$ since $L(z,a)$ is nonvanishing. As we have before, we list these zeroes (with multiplicity) $a_1, a_2, \ldots, a_{n-1}$. When we want to emphasize the dependence of these zeroes on a we will write $a_j = Z_j(a)$. As before, let γ_j, $j = 1, \ldots, n$, denote the boundary curves of Ω.

Theorem 27.1. *Let w_k be a sequence in Ω that tends to a point a in the boundary curve γ_m of Ω. As w_k tends to a, the zeroes $Z_j(w_k)$ of $S(z,w_k)$ become simple zeroes, and it is possible to order them so that for each j, $j \neq m$, there is a point $a_j \in \gamma_j$ such that $Z_j(w_k)$ tends to a_j.*

If a is a point in the boundary of Ω, then $S(z,a)$ is nonvanishing on Ω as a function of z and has exactly $n-1$ zeroes on the boundary of Ω, one on each boundary component not containing a. Furthermore, the zeroes are simple in the sense that $S'(a_j,a) \neq 0$ for each zero a_j. The same is true of $L(z,a)$. In fact, the zeroes of $L(z,a)$ coincide with those of $S(z,a)$.

Proof. If $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between bounded domains with C^∞ smooth boundaries, then the Szegő kernels associated to Ω_1 and Ω_2 transform according to the formula

$$S_1(z,w)^2 = f'(z)S_2(f(z),f(w))^2\overline{f'(w)}. \tag{27.2}$$

Since Ω is biholomorphically equivalent to a domain whose boundary curves are real analytic (via a conformal map that extends to be a C^∞ diffeomorphism of the closures of the domains), and since (27.2) shows that the boundary behavior of the zeroes of the Szegő kernel is conformally invariant, we may assume that Ω is a domain whose boundary curves are real analytic. By Theorem 26.5, such a domain has the virtue that its Szegő kernel extends to be real analytic on a neighborhood of $(\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in b\Omega\}$ and its Garabedian kernel is given by

$$L(z,a) = \frac{1}{2\pi}\frac{1}{z-a} + \ell(z,a),$$

where $\ell(z,a)$ extends to be holomorphic in a neighborhood of $(\overline{\Omega} \times \overline{\Omega})$.

We first prove that the Ahlfors maps $f_{(w_k)}$ tend to a constant of unit modulus as $k \to \infty$. Using formula (7.1), thinking of a as being the point in the boundary and z as being the "other point," we have

$$S(z,a) = -iL(a,z)T(a) = iL(z,a)T(a).$$

We know that neither $S(z,a)$ nor $L(z,a)$ can vanish for z in Ω when $a \in b\Omega$. Consequently, $S(z,a)/L(z,a) = iT(a)$ for $z \in \Omega$. We therefore see that $f_{(w_k)}$ tends uniformly on compact subsets of Ω to the constant function $iT(a)$ as $k \to \infty$. Hence, it follows that the zeroes of $S(z, w_k)$ must tend to the boundary as $k \to \infty$.

Next, we show that $L(z,a)$ has at least one zero on each of the curves γ_j, $j \neq m$. Suppose that $L(z,a)$ is nonvanishing for $z \in \gamma_j$. When both points a and z are in the boundary and $a \neq z$, identity (26.2) yields

$$T(z)L(z,a)T(a) = \overline{L(z,a)}.$$

Let Δ_L^j denote the increase in arg $L(z,a)$ as z traces out γ_j in the standard sense. The last identity reveals that

$$\pm 2\pi + \Delta_L^j = -\Delta_L^j,$$

and, therefore, that $\Delta_L^j = \pm\pi$. But this is impossible; Δ_L^j must be an integer multiple of 2π. Hence, $L(z,a)$ has at least one zero on γ_j. The same is true of $S(z,a)$ because, by virtue of identity (7.1), the zeroes in the z variable of $S(z,a)$ and $L(z,a)$ coincide when $a \in b\Omega$.

Fix a positive integer j, $j \neq m$. Let $R(z)$ denote an antiholomorphic reflection function for γ_j (like the function $1/\bar{z}$ for the unit circle; such functions were constructed in Chapter 11). This reflection function fixes γ_j, is locally diffeomorphic near γ_j, locally maps the outside of Ω to the inside, and $R(R(z)) = z$. Fix a point A in Ω. By (7.1), we may write $S(A,z) = -iL(z,A)T(z)$ and $S(w_k,z) = -iL(z,w_k)T(z)$ for $z \in b\Omega$.