THE DIRICHLET AND NEUMANN AND DIRICHLET-TO-NEUMANN PROBLEMS IN QUADRATURE, DOUBLE QUADRATURE, AND NON-QUADRATURE DOMAINS

STEVEN R. BELL

Abstract. We demonstrate that solving the classical problems mentioned in the title on quadrature domains when the given boundary data is rational is as simple as the method of partial fractions. A by-product of our considerations will be a simple proof that the Dirichlet-to-Neumann map on a double quadrature domain sends rational functions on the boundary to rational functions on the boundary. The results extend to more general domains if rational functions are replaced by the class of functions on the boundary that extend meromorphically to the double.

1. Introduction

It has come to light that double quadrature domains in the plane exist in great abundance and that they can be viewed as replacements for the unit disc when it comes to questions of computational complexity in conformal mapping and potential theory. They are especially useful in the multiply connected setting. The "improved Riemann mapping theorem" described in [10] and further expounded upon in [8] allows one to map domains in the plane, even multiply connected domains, to *nearby* double quadrature domains, thus providing the means to pull back objects on double quadrature domains to the original domain. Double quadrature domains share many of the beautiful and simple properties of the unit disc. The purpose of this paper is to explain methods for solving some of the classical problems of potential theory in quadrature domains that are every bit as simple as similar problems on the unit disc. In fact, the methods will be shown to be analogous to the method of partial fractions from freshman calculus.

To illustrate our motives, consider the Dirichlet problem on the unit disk with polynomial boundary data $P(z,\bar z)$. The most direct way to solve this problem is to use the Schwarz function $S(z)=1/z$ for the unit disc, noting that $\bar z = 1/z$ and $z=1/\bar z$ on the unit circle, replacing terms of the form $z^n\bar z^m$ by $z^{(n-m)}$ if $n\ge m$ and by $\bar z^{(m-n)}$ if $m>n$, to obtain a holomorphic polynomial plus an antiholomorphic polynomial that solves the Dirichlet problem. If the boundary data is a rational function $R(z,\bar z)$, then the solution to the Dirichlet problem can

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be obtained by noting that $R(z, 1/z)$ is a meromorphic function with the same boundary values as the data. If we let $r(z)$ denote the sum of the principal parts of this function inside the unit disk and note that $r(1/\bar{z})$ is an antiholomorphic rational function without poles in the disc that has the same boundary values as $r(z)$, we see that the solution to the Dirichlet problem is the rational function $R(z, 1/z) - r(z) + r(1/\bar{z})$. (See [12] and [9] for the history of these ideas and for other pathways of extending them.) We seek to come up with similar elementary methods in quadrature domains.

In this paper, we will consider problems in potential theory with rational boundary data in two special types of domains. The first type will be bounded quadrature domains with respect to area measure without cusps in the boundary. We will call such domains *area quadrature domains*. The second type will be area quadrature domains which are also quadrature domains with respect to boundary arc length. We will call such domains *double quadrature domains*. We refer the reader to [10] for the precise definitions of these domains and for a summary of their basic properties. The theory of area quadrature domains was pioneered by Aharonov and Shapiro in [1] in the simply connected setting and by Gustafsson [15] in the multiply connected setting. Quadrature domains with respect to boundary arc length were studied by Shapiro and Ullemar in the simply connected setting in [21] and by Gustafsson in [16] in the multiply connected setting. Good references for information about quadrature domains, their usefulness, and the history of the subject are the book [13], the papers [17] and [11] therein, and the book [20].

We now list some of the key properties of area and double quadrature domains that we will need in what follows. To begin, assume that Ω is an area quadrature domain. Then Ω has a boundary consisting of finitely many non-intersecting C^∞ smooth real analytic curves which are, in fact, real algebraic. Gustafsson [15] (after Aharonov and Shapiro [1] in the simply connected case) showed that the Schwarz function $S(z)$ associated to Ω extends meromorphically to the double of Ω. (A meromorphic function h on Ω that extends continuously up to the boundary extends meromorphically to the double if and only if there is a meromorphic function H on Ω that also extends continuously to the boundary such that $h = \overline{H}$ on the boundary.) Since

$$S(z) = \bar{z} \tag{1.1}$$

on the boundary, it follows that z extends meromorphically to the double (by defining the extension to be $\overline{S(z)}$ on the backside) and $S(z)$ extends meromorphically to the double (by defining the extension to be $\bar{z}$ on the backside). Gustafsson showed that the meromorphic extensions of the two functions z and $S(z)$ to the double form a primitive pair for the double, meaning that they generate the field of meromorphic functions on the double. (See Farkas and Kra [14] for the basic facts about primitive pairs and the field of meromorphic functions on the double.) Identity (1.1) allows us to see that a rational function $R(z, \bar{z})$ of z and $\bar{z}$ is equal to $R(z, S(z))$ on the boundary and this yields an extension of the rational function to the double as a meromorphic function. Conversely, if G

is a meromorphic function on the double, then G is a rational combination of the extensions of z and $S(z)$. Since $S(z) = \bar{z}$ on the boundary, we see that the restriction of G to the boundary is rational. We have seen, therefore, that the field of rational functions $R(z, \bar{z})$ on the boundary is precisely the set of meromorphic functions on the double restricted to the boundary. Consequently, if the rational function does not blow up on the boundary, then it is C^∞ smooth on the boundary. Let $\mathcal{R}(b\Omega)$ denote the set of rational functions on the boundary without singularities on the boundary, and let $\mathcal{R}(\Omega)$ denote the set of meromorphic functions on Ω obtained by extending functions in $\mathcal{R}(b\Omega)$ to Ω via the formula $R(z, \bar{z}) = R(z, S(z))$.

The Szegő kernel $S(z, w)$ associated to the area quadrature domain Ω extends holomorphically in z and anti-holomorphically in w to an open set containing $\overline{\Omega} \times \overline{\Omega}$ minus the boundary diagonal. The Garabedian kernel $L(z, w)$ extends holomorphically in z and holomorphically in w to an open set containing $\overline{\Omega} \times \overline{\Omega}$ minus the diagonal. It has a simple pole in the z variable at $z = w$ when $w \in \Omega$ is held fixed. The residue in z at w is $1/2\pi$. The Szegő kernel and Garabedian kernel are non-vanishing in simply connected domains, but on an n-connected domain, the Szegő kernel $S(z, w)$ has $n - 1$ zeroes in z on Ω for each fixed w in Ω. The Garabedian kernel $L(z, w)$, however, is non-zero if $z \in \overline{\Omega}$ and $w \in \Omega$ with $z \neq w$ even in the multiply connected case. If $a \in \Omega$, then neither $S(z, a)$ nor $L(z, a)$ vanish for z in the boundary. See [3] for proofs of all these facts in the spirit of this paper.

Let $S^0(z, w)$ denote $S(z, w)$ and let $S^m(z, w)$ denote $(\partial/\partial\bar{w})^m S(z, w)$. Similarly, let $L^0(z, w)$ denote $L(z, w)$ and let $L^m(z, w)$ denote $(\partial/\partial w)^m L(z, w)$. The *Szegő span* is the complex linear span $\mathcal{S}$ of all functions $h(z)$ of the form $h(z) = S^m(z, a)$ as a ranges over Ω and m ranges over all non-negative integers. The *Garabedian span* is the complex linear span $\mathcal{L}$ of all functions $H(z)$ of the form $H(z) = L^m(z, a)$ as a ranges over Ω and m ranges over all non-negative integers. The *Szegő plus Garabedian span* is the set $\mathcal{S} + \mathcal{L}$ of all sums $h + H$ where h is in the Szegő span and H is in the Garabedian span. We will often shorten our notation by writing $S_a^m(z) = S^m(z, a)$ and $L_a^m(z) = L^m(z, a)$, and we emphasize here that the unadorned $S(z)$ will always stand for the Schwarz function. Note that, because $L(z, a)$ has a singular part that is a non-zero constant times $(z - a)^{-1}$, the singular part of L_a^m is a non-zero constant times $(z - a)^{-(m+1)}$.

To have a proper feeling for the Szegő and Garabedian spans, we mention here that the Szegő span is a dense subspace of the L^2-Hardy space and the Garabedian span is a dense subspace of the orthogonal complement to the L^2-Hardy space in $L^2(b\Omega)$. Hence $\mathcal{S} + \mathcal{L}$ is dense in L^2 of the boundary. Let $A^\infty(\Omega)$ denote the space of holomorphic functions on Ω in $C^\infty(\overline{\Omega})$. The Szegő span is also a dense subspace of $A^\infty(\Omega)$ and the Garabedian span is a dense subspace of the orthogonal complement to the L^2-Hardy space in the topology of $C^\infty(b\Omega)$. Hence $\mathcal{S} + \mathcal{L}$ is dense in C^∞ on the boundary. (See [2] and [3] for proofs of these facts in the more general smooth domain case. The density of the space of rational functions on the boundary of an area quadrature domain in C^∞ of the boundary is also proved in the last section of [8].)

Let $T(z)$ denote the complex unit tangent vector function defined on the boundary of Ω and pointing in the direction of the standard orientation of the boundary. A very important identity at the heart of much of this paper is the relationship between the Szegő kernel and the Garabedian kernel,

$$\overline{S_a(z)} = \frac{1}{i} L_a(z)T(z), \tag{1.2}$$

which holds for $z \in b\Omega$ and $a \in \Omega$. We may differentiate this identity with respect to a to obtain

$$\overline{S_a^m(z)} = \frac{1}{i} L_a^m(z)T(z). \tag{1.3}$$

If Ω is a double quadrature domain, then all the properties above hold plus the property (proved by Gustafsson in [16]) that $T(z)$ extends to the double as a meromorphic function. Consequently, identities (1.2) and (1.3) show that the functions S_a^m and L_a^m also extend to the double for each a in Ω and $m \geq 0$.

With these preliminaries behind us, we can state our main results. We call the first result the *Basic Decomposition.*

Theorem 1.1. *Given a point a in an area quadrature domain Ω,*

$$S_a \mathcal{R}(b\Omega) = \mathcal{S} + \mathcal{L},$$

where the function spaces on the right are understood to be restricted to the boundary. On a double quadrature domain,

$$\mathcal{R}(b\Omega) = \mathcal{S} + \mathcal{L}.$$

We will prove this theorem in the next section, where it will be seen that the coefficients that appear in the decompositions are determined by the principal parts of two meromorphic functions. We remark here that, because the functions on the left hand side of the equalities in Theorem 1.1 extend meromorphically to Ω via the $R(z,\bar{z}) = R(z, S(z))$ substitution, and the functions on the right also extend, we may also state the following result.

Theorem 1.2. *Given a point a in an area quadrature domain Ω,*

$$S_a \mathcal{R}(\Omega) = \mathcal{S} + \mathcal{L}.$$

On a double quadrature domain,

$$\mathcal{R}(\Omega) = \mathcal{S} + \mathcal{L}.$$

We will show how the decomposition in Theorem 1.1 can be used to solve the Dirichlet problem in §3. We will consider the Dirichlet-to-Neumann map in §4, and finally the Neumann problem in §7.

Note that functions in $\mathcal{S}$ are holomorphic on Ω and functions in $\mathcal{L}$ have poles. Hence, it follows as a corollary to Theorem 1.2 that the class of holomorphic functions on a double quadrature domain which extend meromorphically to the double and which have no singularities on the boundary is exactly equal to the Szegő span.

Since $\mathcal{S}+\mathcal{L}$ is an orthogonal sum, the following theorem is an easy consequence of Theorem 1.1.

Theorem 1.3. *Let a be a point in an area quadrature domain Ω. The Szegő projection associated to Ω maps $S_a\mathcal{R}(b\Omega)$ onto the Szegő span. If Ω is a double quadrature domain, it maps $\mathcal{R}(b\Omega)$ onto the Szegő span.*

The results of the next section will therefore show that the Szegő projection of certain functions can be computed via rather straightforward algebra on quadrature domains.

Before we start proving the decompositions, we state that similar and related results hold for the Bergman kernel and span. Let $K(z,w)$ denote the Bergman kernel associated to a bounded area quadrature domain and let $\Lambda(z,w)$ denote the complementary kernel (or conjugate kernel) to the Bergman kernel related to $K(z,w)$ via

$$K(z,a)T(z) = -\overline{\Lambda(z,a)T(z)} \tag{1.4}$$

for a in Ω and z in the boundary. See [3, p. 134] for the definition and basic properties of $\Lambda(z,w)$. We may define the Bergman span $\mathcal{B}$ and the complementary kernel span $\mathit{\Lambda}$ exactly as we defined the Szegő span and Garabedian span. Let $\mathcal{R}'(\Omega)$ denote the set of meromorphic functions on Ω that are derivatives of functions in $\mathcal{R}(\Omega)$.

Theorem 1.4. *On a simply connected area quadrature domain Ω,*

$$\mathcal{R}'(\Omega) = \mathcal{B} + \mathit{\Lambda}.$$

On a multiply connected area quadrature domain, $\mathcal{R}'(\Omega)$ is comprised of the functions in $\mathcal{B} + \mathit{\Lambda}$ with vanishing periods.

We will explain in §5 why a major part of the proof of Theorem 1.4 should be attributed to Björn Gustafsson.

We remark, that since all non-zero functions in the space $\mathit{\Lambda}$ have poles in Ω, a corollary of Theorem 1.4 is that, on an area quadrature domain, a function in $\mathcal{R}'(\Omega)$ without singularities in Ω must be in the Bergman span. This result will allow us to characterize the image of the rational functions under the Dirichlet-to-Neumann map of an area quadrature domain.

We list here a couple important properties that $\mathcal{R}'(\Omega)$ satisfies on an area quadrature domain that we will use throughout the paper. The first is that $\mathcal{R}'(\Omega) \subset \mathcal{R}(\Omega)$. This is because, if h extends to the double on an area quadrature domain, then so does h' (see [6]). Hence, functions in $\mathcal{R}'(\Omega)$ have rational boundary values. Second, it is clear that $\mathcal{R}'(\Omega)$ contains all complex polynomials, and in fact all complex rational functions with residue free poles off the boundary of Ω. It follows from this and our previous remarks that the Bergman span contains all complex polynomials, and in fact all complex rational functions that are holomorphic in a neighborhood of $\overline{\Omega}$ and have only residue free poles outside of $\overline{\Omega}$. (This last fact was proved in [7].)

Call the map that takes Dirichlet problem boundary data to the normal derivative of the solution to the Dirichlet problem the D-to-N map.

Theorem 1.5. *On an area quadrature domain, the D-to-N map takes $\mathcal{R}(b\Omega)$ into $\mathcal{B}T + \overline{\mathcal{B}T}$. If the area quadrature domain is simply connected, the map is onto. If the area quadrature domain is n-connected with $n > 1$, the image of the map is a vector subspace of $\mathcal{B}T + \overline{\mathcal{B}T}$ of codimension $n-1$ described in §6.*

Theorem 1.6. *On a double quadrature domain, $\mathcal{B}T + \overline{\mathcal{B}T}$ is contained in $\mathcal{R}(b\Omega)$ and so the D-to-N map takes $\mathcal{R}(b\Omega)$ into itself.*

We will show that, on an area quadrature domain, functions in $\mathcal{B}T + \overline{\mathcal{B}T}$ uniquely determine the functions in the Bergman span appearing in the sum, as made precise in the following theorem.

Theorem 1.7. *On an area quadrature domain, functions in $\mathcal{B}T + \overline{\mathcal{B}T}$ are represented as $\kappa_1 T + \overline{\kappa_2 T}$ by uniquely determined elements κ_1 and κ_2 in the Bergman span.*

Results like Theorems 1.5 and 1.7 allow one to contemplate in area quadrature domains a one-sided inverse to the D-to-N map on the relevant spaces, especially in the simply connected case. The inverse can be defined in rather explicit terms via algebra. A reader who wanted such a thing could find the details in defining the map by unraveling the proofs of the theorems in §5 and §6.

In the last section of the paper, we consider the solid Dirichlet problem on area quadrature domains, i.e., the problem of, given a function u on the domain, solving $\Delta\varphi = u$ on the domain with $\varphi = 0$ on the boundary. Theorem 1.4 has implications when the function u belongs to various special classes, including polynomials and certain classes of rational and algebraic functions. A new space of functions that I call the *real* Bergman span consisting of all complex linear combinations of $\kappa_1\overline{\kappa_2}$ where κ_1 and κ_2 are in the Bergman span comes into play. This space has many interesting properties. For example, it contains all the real polynomials and is dense in the space of functions that are C^∞ up to the boundary. The Bergman projection maps functions in the real Bergman span onto the Bergman span.

I would like to thank Björn Gustafsson here for reading an earlier version of this paper. His helpful remarks led me to correct, strengthen, and improve some of the theorems and to improve the organization of the paper.

2. The basic decomposition in area quadrature domains

Suppose that Ω is a bounded area quadrature domain and suppose $\phi(z) = R(z, \bar{z})$ is a rational function of z and $\bar{z}$ without singularities on $b\Omega$. We will now explain how to produce a finite decomposition of ϕ on the boundary in terms of the Szegő kernel and the Garabedian kernel that can be thought of as an analogue of a "partial fractions decomposition" on the boundary. The decomposition will allow us to solve the Dirichlet problem with rational boundary data in finite terms.

Pick a point a in Ω. Notice that

$$S_a(z)\phi(z) = S_a(z)R(z,\bar{z}) = S_a(z)R(z,S(z))$$

on the boundary, and this defines a meromorphic extension G of $S_a\phi$ to Ω. We now subtract off the unique linear combination λ of the functions of the form $L_{b_k}^m$ to remove the poles of the meromorphic function G on Ω. We will show that

$$h(z) := G(z) - \lambda(z)$$

is a function in the Szegő span $\mathcal{S}$. Indeed, if we pair h with a function g in the dense subset of the Hardy space consisting of functions in $A^\infty(\Omega)$, and note that functions of the form $L_{b_k}^m$ are orthogonal to the Hardy space, we may use the identity $S(z) = \bar{z}$ and identity (1.2) to see that

$$\langle g, h\rangle = \int_{b\Omega} g(z)\,\overline{S_a(z)R(z,S(z))}\,ds = -i\int_{b\Omega} g(z)L_a(z)\,\overline{R(\,\overline{S(z)}\,,\bar{z})}\,dz,$$

and the residue theorem shows that this last integral yields a finite linear combination of values of g and its derivatives at finitely many points. Hence, h is equal to the linear combination of the functions $S_{a_k}^m$ which would have the same effect when paired with g.

We have shown that there are finitely many points a_n and b_n in Ω and positive integers N_S, M_S, N_L, and M_L such that

$$S_a(z)R(z,\bar{z}) = \sum_{n=1}^{N_S}\sum_{m=0}^{M_S} A_{nm}S_{a_n}^m(z) + \sum_{n=1}^{N_L}\sum_{m=0}^{M_L} B_{nm}L_{b_n}^m(z) \tag{2.1}$$

on the boundary of Ω.

We remark here that, just as in the method of undetermined coefficients, the coefficients and points in the decomposition (2.1) are uniquely determined by the principal parts of meromorphic functions with finitely many poles. Indeed the coefficients B_{nm} and the points b_n were chosen so that the principal parts of the sum λ match the principal parts of $S_a(z)R(z,S(z))$. If we multiply the decomposition by $T(z)$ and use identities (1.2) and (1.3), and note that $R(z,\bar{z}) = R(\,\overline{S(z)}\,,\bar{z})$ on the boundary, we obtain after conjugation

$$L_a(z)\overline{R(\,\overline{S(z)}\,,\bar{z})} = \sum_{n=1}^{N_S}\sum_{m=0}^{M_S} \overline{A_{nm}}\,L_{a_n}^m(z) + \sum_{n=1}^{N_L}\sum_{m=0}^{M_L} \overline{B_{nm}}\,S_{b_n}^m(z), \tag{2.2}$$

and so we see that the coefficients A_{nm} and the points a_n are determined by the principal parts of $L_a(z)\overline{R(\,\overline{S(z)}\,,\bar{z})}$ in Ω.

We have shown that S_a times a rational function is in the Szegő plus Garabedian span when restricted to the boundary. To finish the proof of Theorem 1.1 for area quadrature domains, we need to show that a function in $\mathcal{S}+\mathcal{L}$ divided by S_a, when restricted to the boundary, is rational and without singularities on the boundary. If we divide a function in $\mathcal{S}+\mathcal{L}$ by S_a, we obtain a sum of functions of the form $S_{a_n}^m/S_a$ and $L_{a_n}^m/S_a$. Such functions extend to the double of Ω as meromorphic functions because identities (1.2) and (1.3) show that they are equal to the conjugates of $L_{a_n}^m/L_a$ and $S_{a_n}^m/L_a$, respectively, on the boundary. Since these functions extend to the double and do not have singularities on the boundary, they are therefore rational combinations of z and $S(z)$, which when restricted to the boundary, are rational functions of z and $\bar{z}$ without singularities on the boundary. This completes the proof of the part of Theorem 1.1 about area quadrature domains.

We remark that the space of functions on the boundary given by S_a times a rational function is easily seen to be independent of the point a since quotients of the form S_a/S_b extend meromorphically to the double, and are therefore rational on the boundary.

On a double quadrature domain, S_a extends to the double as a meromorphic function and has no singularities on the boundary (see [10]). Therefore $S_a\mathcal{R}=\mathcal{R}$ and the proof of Theorem 1.1 is complete.

3. Using the decomposition to solve the Dirichlet problem

We first assume that Ω is a *simply connected* area quadrature domain. We will now show how the decomposition (2.1) produces a simple and explicit solution to the Dirichlet problem with rational boundary data $R(z,\bar{z})$. Recall that S_a and L_a are non-vanishing on $\overline{\Omega}$ and extend holomorphically past the boundary in case Ω is simply connected. Notice that if we divide the decomposition (2.1) by S_a, then we decompose our boundary data $R(z,\bar{z})$ into a finite sum of functions of the form S_b^m/S_a and L_b^m/S_a. The functions S_b^m/S_a extend holomorphically to Ω and are smooth up to the boundary. Identities (1.2) and (1.3) reveal that L_b^m/S_a is equal to the conjugate of S_b^m/L_a on the boundary and therefore these functions extend antiholomorphically to Ω and are smooth up to the boundary. (Note that the simple pole of L_a at a gives rise to a zero of the quotient at a.) Consequently, we can read off the conclusion of the following theorem.

Theorem 3.1. *The solution to the Dirichlet problem on a simply connected area quadrature domain with rational boundary data $R(z,\bar{z})$ can be read off from the basic decomposition of $S_a(z)R(z,\bar{z})$ and is of the form $h+\overline{H}$ where both h and H are sums of quotients that extend holomorphically to Ω and meromorphically to the double. In fact, h is a quotient of the form σ_1/S_a and H is a quotient of the form σ_2/L_a where σ_j, $j=1,2$, are functions in the Szegő span.*

Note that meromorphic functions on the double are generated by the meromorphic extensions of z and $S(z)$ to the double, and that $S(z)$ is algebraic. Thus, we have given an alternate way of looking at Ebenfelt's theorem [12] about the algebraicity of the solution to the Dirichlet problem with rational boundary data on simply connected area quadrature domains (see also [9] for more about this fascinating subject).

We may repeat much of the same reasoning in case Ω is an n-connected area quadrature domain, taking into account that S_a has $n-1$ zeroes. We may choose the point a so that the $n-1$ zeroes of S_a are distinct and simple (see [3, p. 105]). Let $a_1, \ldots, a_{n-1}$ denote these zeroes. Let $G(z,w)$ denote the classical Green's function associated to Ω and write $G_w^0(z) = G(z,w)$. For $m \geq 1$, define

$$G_w^m(z) := \frac{\partial^m}{\partial w^m} G(z,w) \quad \text{and} \quad G_w^{\bar{m}}(z) := \frac{\partial^m}{\partial \bar{w}^m} G(z,w).$$

Note that, as a function of z, $G_b^m(z)$ has a singular part at $b \in \Omega$ that is a non-zero constant times $1/(z-b)^m$, is harmonic in z on $\Omega - \{b\}$, extends continuously to the boundary and vanishes on the boundary. Similarly, $G_b^{\bar{m}}(z)$ has a singular part that is a non-zero constant times the conjugate of $1/(z-b)^m$, is harmonic in z on $\Omega - \{b\}$, extends continuously to the boundary and vanishes on the boundary.

To solve the Dirichlet problem on Ω, given rational boundary data $R(z, \bar{z})$, as in the simply connected case, the decomposition (2.1) yields

$$R(z, \bar{z}) = h + \overline{H}$$

on the boundary, where

$$h = \frac{\sigma_1(z)}{S_a(z)} \qquad \text{and} \qquad H = \frac{\sigma_2(z)}{L_a(z)},$$

and where σ_1 and σ_2 are in the Szegő span. Note that h and H extend smoothly to the boundary, that H is holomorphic on Ω because L_a is non-vanishing on $\overline{\Omega} - \{a\}$ and the pole of L_a creates a zero of H at a, but h is perhaps only meromorphic since it may have simple poles at some or all of the zeroes of S_a. However, by subtracting off appropriate constants times $G_{a_k}^1$ for each of the zeroes a_k to remove the simple poles, noting that these functions vanish on the boundary, we obtain the harmonic extension of our boundary data to Ω in the form

$$h + \overline{H} + \sum_{k=1}^{n-1} c_k G_{a_k}^1, \tag{3.1}$$

where h and H extend meromorphically to the double, and hence are rational combinations of z and the Schwarz function $S(z)$. Consequently, we have an alternate way to that given in [8] to see that the solution to the Dirichlet problem with rational boundary data is algebraic, modulo an $n-1$ dimensional subspace.

4. The Dirichlet-to-Neumann map

We now continue the line of thought of the last section and consider the implications for the D-to-N map for rational boundary data on an area quadrature domain Ω. It is shown in [3, p. 134-135] that, if $w \in \Omega$ is held fixed, the normal derivative $(\partial/\partial n_z)$ of $G_w^0(z)$ with respect to z is given by

$$\frac{\partial}{\partial n_z} G_w^0(z) = -iT(z) \frac{\partial}{\partial z} G_w^0(z) + i\, \overline{T(z)}\, \overline{\frac{\partial}{\partial z} G_w^0(z)}.$$

We remark here that it is also shown there that this expression can be further manipulated to yield two more expressions for the same normal derivative,

$$\frac{\partial}{\partial n_z} G_w^0(z) = -2iT(z)\frac{\partial}{\partial z} G_w^0(z) = 2i\,\overline{T(z)}\,\overline{\frac{\partial}{\partial z} G_w^0(z)}.$$

Although these expressions are shorter and simpler, we will have reason to prefer the longer form. Hence, for $m \geq 1$, the normal derivative of $G_w^m(z)$ is given by

$$\frac{\partial}{\partial n_z} G_w^m(z) = -iT(z)\frac{\partial}{\partial z} G_w^m(z) + i\,\overline{T(z)}\,\overline{\frac{\partial}{\partial z} G_w^{\bar{m}}(z)}, \tag{4.1}$$

and the alternative expressions above yield

$$\frac{\partial}{\partial n_z} G_w^m(z) = -2iT(z)\frac{\partial}{\partial z} G_w^m(z) = 2i\,\overline{T(z)}\,\overline{\frac{\partial}{\partial z} G_w^{\bar{m}}(z)}.$$

It is shown in [3, p. 77, 134-135] that the normal derivative of the solution to the Dirichlet problem given by equation (3.1) is

$$-ih'(z)T(z) + i\,\overline{H'(z)T(z)} - iT(z)\sum_{k=1}^{n-1} c_k \frac{\partial}{\partial z} G_{a_k}^1(z) + i\overline{T(z)}\sum_{k=1}^{n-1} c_k \overline{\frac{\partial}{\partial z} G_{a_k}^{\bar{1}}(z)}.$$

But, for $w \in \Omega$, the Bergman kernel $K(z,w)$ is related to the Green's function via

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial \bar{w}} G(z,w)$$

and the complementary kernel $\Lambda(z,w)$ to the Bergman kernel is given by definition as

$$\Lambda(z,w) = -\frac{2}{\pi}\frac{\partial^2}{\partial z \partial w} G(z,w).$$

(See [3, p. 134] for these identities and the basic properties of $\Lambda(z,w)$.) Consequently, the last formula for the normal derivative can be rewritten as

$$-ih'(z)T(z) + i\,\overline{H'(z)T(z)} + \frac{\pi i}{2} T(z)\sum_{k=1}^{n-1} c_k \Lambda(z,a_k) - \frac{\pi i}{2}\,\overline{T(z)}\,\sum_{k=1}^{n-1} c_k \overline{K(z,a_k)}. \tag{4.2}$$

It is shown in [6] that, on an area quadrature domain, if a meromorphic function g extends meromorphically to the double, then g' also extends meromorphically to the double. It is also shown in [6] that the Bergman kernel extends meromorphically to the double on an area quadrature domain. We will prove momentarily that $\Lambda(z,a_k)$ also extends meromorphically to the double as a function of z. Hence, we have expressed the normal derivative of the solution to the Dirichlet problem as $gT + \overline{GT}$ where g and G are meromorphic functions on Ω that extend meromorphically to the double, and are consequently rational combinations of z and $S(z)$. When we restrict to the boundary, we conclude that g and G are rational functions of z and $\bar{z}$ on the boundary. It is shown in [6] that, on an area quadrature domain, the function T^2 extends to the double as a meromorphic function. Hence, the Neumann boundary data of the solution to the Dirichlet problem with rational boundary data is a sum of a rational function times the square root of a rational function plus the conjugate of such expressions. On a double quadrature domain, the function T itself extends meromorphically to the double, and in this case, we may state that the D-to-N map sends rational functions of z and $\bar{z}$ to rational functions of z and $\bar{z}$. In the next section, we determine exactly which functions appear in the range of the D-to-N map.

We now give the promised proof that $\Lambda(z,a)$ extends meromorphically in z to the double for fixed $a \in \Omega$ when Ω is an area quadrature domain. Equation (1.4) shows that $\Lambda(z,a)$ is equal to minus the conjugate of the quantity $K(z,a)$ times $T(z)^2$ on the boundary. Since both of these functions extend meromorphically to the double, and are therefore rational functions of z and $S(z) = \bar{z}$ on the boundary, it follows that $\Lambda(z,a)$ extends meromorphically to the double and is equal to a rational combination of z and $\bar{z}$ on the boundary.

The identity (1.4) for the simpler expressions for the normal derivatives of the Green's functions could be used to simplify (4.2) to read

$$-ih'(z)T(z) + i\,\overline{H'(z)T(z)} + \pi i T(z)\sum_{k=1}^{n-1} c_k \Lambda(z,a_k), \tag{4.3}$$

or even

$$-ih'(z)T(z) + i\,\overline{H'(z)T(z)} - \pi i\,\overline{T(z)}\sum_{k=1}^{n-1} c_k \overline{K(z,a_k)}, \tag{4.4}$$

but we prefer (4.2) because the poles of the $\Lambda(z,a_k)$ terms exactly cancel the poles of h' the way we have chosen the coefficients, and therefore the normal derivative is in fact expressed as $gT + \overline{GT}$ where g and G are *holomorphic* functions on Ω that extend smoothly to the boundary, and that extend meromorphically to the double. We will have more to say on this subject later when we prove Theorem 1.5.

An interesting consequence of equations (4.3) and (4.4) is that they imply that certain period matrices are non-singular.

Theorem 4.1. *Suppose that the zeroes $a_1, \dots a_{n-1}$ of the Szegő kernel associated to a point a in an n-connected area quadrature domain Ω are simple. Then the matrix of periods associated to the functions $K(z,a_k)$, $k = 1, \dots, n-1$ is non-singular. So is the matrix of periods associated to the functions $\Lambda(z,a_k)$, $k = 1, \dots, n-1$.*

We remark that it was proved in [7] that, if Ω is n connected, then there *exist* $n-1$ points $b_1, b_2, \dots, b_{n-1}$ in Ω such that the period matrix of $K(z,b_k)$ is non-singular. It is also interesting to note that, because of the way the zeroes of the Szegő kernel and the periods of the functions in Theorem 4.1 transform under conformal changes of variables, and because smoothly bounded n-connected domains are conformaly equivalent to an area quadrature domain via Gustafsson's theorem [15], Theorem 4.1 can be seen to hold for general bounded smooth n-connected domains as well.

We now prove Theorem 4.1. As is standard, let ω_k denote the harmonic measure function which is harmonic on Ω and equal to one on the k-th boundary curve of the $n-1$ inner boundary curves and equal to zero on the other boundary curves, and let F_k' denote $2(\partial/\partial z)\omega_k$. Note that because the rational functions are dense in $C^\infty(b\Omega)$, we may approximate each ω_k on $b\Omega$ by rational functions r_k. Let R_k denote the harmonic function on Ω with boundary values given by r_k. The normal derivative of R_k is given by equation (4.3), and it can be made as close in $C^\infty(b\Omega)$ to the normal derivative $-iF_k'T$ of ω_k as desired (see [3, p. 87] for the calculation of this normal derivative). Since the periods of F_k', $k = 1, \dots, n-1$ are well known to be linearly independent, and since the periods of the functions given by equation (4.3) are linear combinations of the periods of $\Lambda(z, a_k)$, it follows that the periods of $\Lambda(z, a_k)$ are independent. Identity (1.4) shows that the periods of $K(z, a_k)$ are just minus the conjugates of the periods of $\Lambda(z, a_k)$, and so it follows that the periods of $K(z, a_k)$ are also independent. This completes the proof of Theorem 4.1. (We remark here that we could have found other suitable functions to serve as the R_k by approximating $\ln|z - b_k|$ where b_k is a point inside the k-th inner hole of Ω instead of ω_k.)

We will need the harmonic functions R_k constructed in the proof above later in the paper. The key properties that we will need later are that these harmonic functions have boundary values given by rational functions and the periods of the normal derivatives of the R_k, $k = 1, \dots, n-1$, are independent.

5. Proofs of the Bergman span results

The proof of Theorem 1.4 follows a similar pattern to the arguments in the last section and is motivated by a result of Gustafsson stated as Lemma 4 in [5]. In fact, there are two proofs of a closely related theorem given in [5] and a statement without proof of a converse that is relevant here. The proof we set out below is a third way of looking at this problem, and is shorter and simpler than the arguments given in [5], but it must be said that the meat of the argument is Gustafsson's idea (inspired by an older result of Schiffer and Spencer [19]).

Suppose that Ω is an area quadrature domain and that h is a function in $\mathcal{R}(\Omega)$. It follows that h' has only finitely many poles in Ω which are residue free poles of order two or more. Note that the singular part of $\Lambda(z, a)$ is equal to a non-zero constant times $(z-a)^{-2}$, and the singular part of $\Lambda^m(z, a)$ is equal to a non-zero constant times $(z - a)^{-(m+2)}$. Hence, there is an element λ in the span $\mathit{\Lambda}$ that has the same principal parts at each of the poles of h'. But such a function λ is given as the derivative $(\partial/\partial z)$ of a finite sum of the form

$$\phi(z) = \sum_{n,m} c_{nm} G^m_{a_n}(z).$$

Note that ϕ vanishes on the boundary and that $h-\phi$ also has removable singularities at the poles of h. Also note, that because h extends meromorphically to the double, there is a meromorphic function H on Ω that extends smoothly to the boundary (and without singularities on the boundary) such that $h(z) = \overline{H(z)}$ for $z \in b\Omega$. Now, given g in $A^\infty(\Omega)$, we may compute the inner product $\langle h' - \lambda, g \rangle_\Omega =$

$$\frac{i}{2}\int_\Omega (h' - \lambda)\,\bar{g}\;dz \wedge d\bar{z} = \frac{i}{2}\int_{b\Omega} (h - \phi)\bar{g}\;d\bar{z} = \frac{i}{2}\int_{b\Omega} h\,\bar{g}\;d\bar{z} = \frac{i}{2}\int_{b\Omega} \bar{H}\,\bar{g}\;d\bar{z},$$

and the residue theorem implies that this last integral is equal to the complex conjugate of a finite linear combination of values of g and its derivatives at finitely many points in Ω. There is an element κ in the Bergman span that has the same effect when paired with g in the $L^2(\Omega)$ inner product. Hence, since $A^\infty(\Omega)$ is dense in the Bergman space, $h' - \lambda = \kappa$ and we have proved that $\mathcal{R}'(\Omega) \subset \mathcal{B} + \mathit{\Lambda}$.

To prove the reverse inclusion, we will need to define some terminology. Note that we may differentiate identity (1.4) with respect to $\bar{a}$ to obtain the identity

$$K^m(z,a)T(z) = -\overline{\Lambda^m(z,a)T(z)} \tag{5.1}$$

for z in the boundary (where the superscript m indicate derivatives of order m with respect to $\bar{a}$ in the Bergman kernel and with respect to a in the Λ-kernel). Given a function $g = \kappa + \lambda$ in $\mathcal{B} + \mathit{\Lambda}$, we define the *complementary function* G to g to be the function gotten by conjugating all the constants in the linear combination, by changing terms of the form K_b^m in g to Λ_b^m in the complementary function, and terms of the form Λ_b^m to K_b^m. In this way, we obtain a function G in $\mathcal{B} + \mathit{\Lambda}$ that satisfies

$$g(z)T(z) = -\overline{G(z)T(z)}$$

on the boundary. If γ is a curve in the boundary of Ω, then identity (5.1) shows that

$$\int_\gamma K^m(z,a)\;dz = -\int_\gamma \overline{\Lambda^m(z,a)}\;d\bar{z},$$

and similarly for integrals of $\Lambda^m(z,a)$ by taking conjugates. Hence, g and G satisfy

$$\int_\gamma g(z)\;dz = -\int_\gamma \overline{G(z)}\;d\bar{z}. \tag{5.2}$$

Hence, if a period of g vanishes, then so does the same period of G.

First, we will prove the reverse inclusion in case Ω is a *simply connected* area quadrature domain. Note that, in this case, elements of $\mathcal{B} + \mathit{\Lambda}$ have single valued antiderivatives on Ω which are meromorphic on Ω because all the poles of elements of $\mathit{\Lambda}$ are residue free and are of order two or more. Let h be such an antiderivative, and write $h' = \kappa + \lambda$ as we did above. Let G be the complementary function to h' and let H be an antiderivative of G. Let γ_z denote a curve that starts at a boundary point b and moves along the boundary to another point z in the boundary. The formula (5.2) holds for the curve γ_z for our complementary functions $g = h'$ and $G = H'$. Hence, $h(z) - h(b)$ is equal to the conjugate of $-(H(z) - H(b))$ on the boundary. This shows that h extends to the double, and the proof of the reverse inclusion is complete in the simply connected case. This completes the proof of Theorem 1.4 in case Ω is simply connected.

We now turn to the multiply connected case. If $\kappa+\lambda$ is an element of $\mathcal{B}+\mathit{\Lambda}$ with vanishing periods, then the periods of the complementary function are also zero and we obtain two meromorphic functions h and H as we did above such that $h'=\kappa+\lambda$ and $h'T=-\overline{H'T}$ on the boundary. Our task now is to show that we may adjust h and H by constants in order to make $h=-\overline{H}$ on the boundary so that we may conclude that h extends to the double. Choose a point b in the outer boundary of Ω and adjust h and H by subtracting off constants $h(b)$ and $H(b)$ from h and H so that $h(b)=0$ and $H(b)=0$. The calculation of the last paragraph shows $h=-\overline{H}$ on the outer boundary. The key to seeing that this identity persists on the inner boundaries is that integrals from b to a point b' on an inner boundary also agree because of the relationships between the kernels and the Green's function. Indeed, if γ is a curve in Ω that starts at b and goes to b' and $w\in\Omega$ is not in γ, then it is well known that

$$\int_\gamma \frac{\partial}{\partial z}G(z,w)\ dz=-\int_\gamma \frac{\partial}{\partial \bar z}G(z,w)\ d\bar z. \tag{5.3}$$

To see this, note that if ϕ is real valued, then

$$\int_\gamma \frac{\partial \phi}{\partial z}\ dz=\frac{1}{2}\int_\gamma \left(\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}\right)(dx+i\,dy),$$

and the real part of this integral is $\phi(b')-\phi(b)$. Hence, since the Green's function is real and vanishes on the boundary, the real parts of the two integrals in (5.3) vanish, and therefore, since they are conjugates of each other, the identity follows. Now differentiating (5.3) with respect to $\bar w$ and letting $w=a$ yields that

$$\int_\gamma K^m(z,a)\ dz=-\int_\gamma \overline{\Lambda^m(z,a)}\ d\bar z.$$

(This is just a reformulation of a well known fact going back to Schiffer and Spencer [19] about the vanishing of the β-periods of the meromorphic differentials obtained by extending $K_a dz$ to the backside of the double as the conjugate of $-\Lambda_a dz$.) To continue, if we now choose such a curve γ from b to b' that avoids points where h' and H' have singularities, this identity shows that

$$h(b')=\int_\gamma h'\,dz=-\int_\gamma \overline{H'}\,d\bar z=-\overline{H(b')}.$$

Consequently, the identity $h=-\overline{H}$ extends to the inner boundary containing b'. We may conclude that h extends to the double as a meromorphic function. This completes the proof of Theorem 1.4.

We remark here that Theorem 4.1 yields that if Ω is n connected, then there exist $n-1$ points $a_1,a_2,\dots,a_{n-1}$ in Ω such that the period matrix of $K(z,a_k)$ is non-singular. Hence, given an element $\kappa+\lambda$ of $\mathcal{B}+\mathit{\Lambda}$, it is possible to subtract off a linear combination of the functions $K(z,a_k)$ so as to make the periods vanish. Hence, Theorem 1.4 yields that $\kappa+\lambda$ is equal to a function in $\mathcal{R}'(\Omega)$ modulo a linear combination of $K(z,a_k)$. If we let $\mathcal{B}_{n-1}$ denote the complex linear span of the $K(z, a_k)$, then we can state that

$$\mathcal{R}'(\Omega) \subset \mathcal{B} + \Lambda \subset \mathcal{R}'(\Omega) + \mathcal{B}_{n-1}.$$

We now turn to the proof of Theorem 1.5. First, we will use Theorem 1.4 to determine the image of $\mathcal{R}(b\Omega)$ under the Dirichlet-to-Neumann map in an area quadrature domain Ω. Formula (4.2) combined with Theorem 1.4 shows that functions in the image are of the form $\kappa_1 T + \lambda_1 T + \overline{\kappa_2 T} + \overline{\lambda_2 T}$ where κ_1 and κ_2 are in $\mathcal{B}$ and λ_1 and λ_2 are in Λ. Identity (5.1) shows how to convert the term $\lambda_1 T$ into a term of the form $\overline{\kappa_3 T}$ where $\kappa_3 \in \mathcal{B}$, and the term $\overline{\lambda_2 T}$ into a term of the form $\kappa_4 T$ where $\kappa_4 \in \mathcal{B}$. Hence, when everything is combined, we obtain an expression in $\mathcal{B}T + \overline{\mathcal{B}T}$. Thus, the D-to-N map maps rational functions into the desired space.

To finish the proof of Theorem 1.5 in the simply connected case, we must show that every element in $\mathcal{B}T + \overline{\mathcal{B}T}$ is equal to the normal derivative of a harmonic function with rational boundary values. Indeed, given a function $\psi = \kappa_1 T + \overline{\kappa_2 T}$ in this space, we may find functions holomorphic functions h_1 and h_2 on Ω such that $-ih_1' = \kappa_1$ and $-ih_2' = \kappa_2$ where, by Theorem 1.4, h_1 and h_2 are in $\mathcal{R}(\Omega)$. We may now write

$$\psi = -ih_1' T + i\,\overline{h_2' T}.$$

Such a function is the normal derivative of the harmonic function with rational boundary data $h_1 + \overline{h_2}$. This completes the proof in the simply connected case.

We will treat the multiply connected case in the next section.

Finally, we will prove Theorem 1.7. We need to show that a representation of the form $\psi = \kappa_1 T + \overline{\kappa_2 T}$ is unique. If such an expression were equal to zero on the boundary, then $\kappa_1 T = -\overline{\kappa_2 T}$ and the left hand side of this expression is orthogonal to holomorphic functions in $L^2(b\Omega)$ and the right hand side is orthogonal to the conjugates of holomorphic functions in $L^2(b\Omega)$. Such functions must be given by linear combinations of $F_k' T$ where F_k' are the well known holomorphic functions that arise via $F_k' = 2(\partial/\partial z)\omega_k$ where ω_k are the harmonic measure functions associated to the $n-1$ inner boundary curves. See [18] or ([3, p. 80]) for a proof of this old result of M. Schiffer. Hence $\kappa_1 = \sum_{k=1}^{n-1} c_k F_k'$. We know that κ_1 is $(\partial/\partial z)$ of a linear combination $\mathcal{G}$ of derivatives of Green's functions of the form $G_b^{\bar{m}}(z)$ as b ranges over a finite set in Ω. Hence, there are constants c_k so that $\mathcal{G} - \sum_{k=1}^{n-1} c_k \omega_k$ is antimeromorphic on Ω. But since this function is continuous up to the outer boundary and zero there, this antimeromorphic function must be zero. Since $\mathcal{G}$ is continuous up to the boundary and vanishes there, it follows that all the constants c_k must be zero. Thus, we have shown that $\mathcal{G}$ must be zero, and so κ_1 must be zero. We also conclude that κ_2 must be zero. Finally, it is an easy exercise to see that if

$$0 \equiv \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm} K^m(z, b_n),$$

then all the coefficients c_{nm} must be zero. (Indeed, such a function would be orthogonal to the Bergman space, and hence orthogonal to all polynomials. However, pairing a polynomial with the function in the Bergman span would yield a non-trivial sum of values and derivatives of the polynomial at finitely many points in Ω, and it is easy to construct a polynomial that would make this sum non-zero, yielding a contradiction.) We have shown that the representation of a function in $\mathcal{B}T + \overline{\mathcal{B}T}$ is uniquely determined.

The techniques of this section allow us to construct a one-sided inverse to the D-to-N map on rational functions in a simply connected area quadrature domain. Indeed, given a basic term like K_a^m in the Bergman span we may express a meromorphic antiderivative $-ih$ of K_a^m on Ω via a path integral formula. The proof of Theorem 1.4 reveals that h is in $\mathcal{R}(\Omega)$. Now, the normal derivative of the solution to the Dirichlet problem with boundary data $h \in \mathcal{R}(b\Omega)$ is equal to $K_a^m T$. By this means, we may define a linear transformation L that maps $K_a^m T$ to the boundary values of h. The same procedure works for conjugates of terms of the form $K_a^m T$. Since the representation of functions in $\mathcal{B}T + \overline{\mathcal{B}T}$ is unique, we obtain an operator L that maps $\mathcal{B}T + \overline{\mathcal{B}T}$ onto $\mathcal{R}(b\Omega)$ such that the D-to-N map composed with L is the identity.

6. The image of the D-to-N map in the multiply connected setting

We now turn to completing the proof of Theorem 1.5 in the n-connected quadrature domain case where $n > 1$. Let $\mathcal{V}$ denote the complex vector space $\mathcal{B}T+\overline{\mathcal{B}T}$ and let $\mathcal{N}$ denote the subspace of $\mathcal{V}$ that is the image of $\mathcal{R}(b\Omega)$ under the D-to-N map. Define a linear equivalence relation $\sim$ on $\mathcal{V}$ as follows. Given $v_j = \alpha_j T + \overline{\beta_j T}$ in $\mathcal{V}$, $j = 1, 2$, we say that $v_1 \sim v_2$ if $\alpha_1 - \alpha_2 = h'$ and $\beta_1 - \beta_2 = H'$ where h and H are holomorphic functions on Ω in $\mathcal{R}(\Omega)$. Note that $v_1 - v_2$ is the normal derivative of $ih - i\overline{H}$ on the boundary. It is easy to verify that $\sim$ is an equivalence relation.

We know that if φ is a harmonic function on Ω with boundary values in $\mathcal{R}(b\Omega)$, then the normal derivative $(\partial\varphi/\partial n)$ is in $\mathcal{V}$. We now claim that, if $v_j = \partial\varphi_j/\partial n$ for two such harmonic functions φ_j, $j = 1, 2$, then $v_1 \sim v_2$ if and only if the normal derivative of $\varphi = \varphi_1 - \varphi_2$ has vanishing periods. (Note that the periods of the harmonic function φ are given by integrals of $\partial\varphi/\partial n$ about the inner boundary curves with respect to arc length measure ds, but the periods of an analytic function are integrals with respect to dz.) Indeed, the normal derivative of a function with boundary values in $\mathcal{R}(b\Omega)$ is given by equation (4.4) and if the periods vanish, then Theorem 4.1 implies that all the coefficients c_k must vanish and the functions h and H have no poles in Ω. Hence, $\varphi = h + \overline{H}$ and $v_1 - v_2 = -ih'T + i\overline{H'T}$ yields that $v_1 \sim v_2$. The reverse implication is automatic.

Note also that if φ is a harmonic function on Ω with boundary values in $\mathcal{R}(b\Omega)$ and $v \in \mathcal{V}$ is such that $(\partial\varphi/\partial n) \sim v$, then v is also equal to the normal derivative of a harmonic function with boundary values in $\mathcal{R}(b\Omega)$.

Let $[\mathcal{V}]$ denote the vector space obtained from $\mathcal{V}$ via the $\sim$ equivalence relation. We now show that $[\mathcal{V}]$ is $2(n-1)$ dimensional by finding explicit basis vectors for the space. Let the points a_k, $k = 1, \ldots, n-1$, be defined as in Theorem 4.1 and let K_{a_k} denote the function $K(z, a_k)$. We claim that the $2(n-1)$ functions $[K_{a_k}T]$ and $[\,\overline{K_{a_k}T}\,]$, for $k = 1, \ldots, n-1$, form a basis for $[\mathcal{V}]$. Indeed, if a linear combination of such vectors is in the zero equivalence class, then an identity of the form

$$\sum_{k=1}^{n-1} \alpha_k K_{a_k} T + \sum_{k=1}^{n-1} \overline{\beta_k K_{a_k} T} = h'T + \overline{H'T} \tag{6.1}$$

would hold, where h and H are holomorphic on Ω and have boundary values in $\mathcal{R}(b\Omega)$. Note that, in this case, h and H both extend meromorphically to the double, and hence there are meromorphic functions g and G on Ω such that $h = \overline{g}$ and $H = \overline{G}$ on the boundary. It is an old result of Schiffer that if f and F are analytic on Ω and $fT = \overline{FT}$ on the boundary, then f and F are linear combinations of the functions $F_k' = 2(\partial/\partial z)\omega_k$ described in §4 (see Theorem 19.1 in [3, p. 80]). Hence, there are coefficients c_k such that

$$h' - \sum_{k=1}^{n-1} \alpha_k K_{a_k} = \frac{1}{2} \sum_{k=1}^{n-1} c_k F_k', \tag{6.2}$$

and this implies that $(\partial/\partial z)$ of

$$h + \frac{2}{\pi} \sum_{k=1}^{n-1} \alpha_k G_{a_k}^{\bar{1}} - \sum_{k=1}^{n-1} c_k \omega_k$$

is equal to zero, i.e., that this function is antimeromorphic. But this function is equal to h on the outer boundary curve, which is also equal to $\overline{g}$ there. This implies that $\overline{g}$ is equal to this antimeromorphic function. Since $h = \overline{g}$ on the inner boundary curves too, we must conclude that all the c_k are zero, and this in turn implies that all the α_k must vanish (in order to make the periods of the function on the left hand side of (6.2) equal to zero). Hence h' is the zero function. Similarly, H' is the zero function and we have shown that the linear combination on the left hand side of (6.1) is equal to zero. Now Theorem 1.7 shows that all the α_k and β_k must be zero and we have shown that the vectors are linearly independent. Finally, we must show that the vectors span $[\mathcal{V}]$.

Given $v = \kappa_1 T + \overline{\kappa_2 T}$ where κ_j, $j = 1, 2$, are in the Bergman span, Theorem 4.1 yields functions k_j in the linear span of $K(z, a_k)$, $k = 1, \ldots, n-1$ such that k_j has the same periods as κ_j, $j = 1, 2$. Now $(\kappa_1 - k_1)T + \overline{(\kappa_2 - k_2)T}$ is equal to $h'T + \overline{H'T}$ where h and H are analytic on Ω with boundary values in $\mathcal{R}(b\Omega)$, and hence v is in the same equivalence class as $k_1 T + \overline{k_2 T}$, and this shows that the vectors span $[\mathcal{V}]$, and therefore form a basis.

Let R_k, $k = 1, \ldots, n-1$ denote the functions defined in the proof of Theorem 4.1 in §4. We next claim that $[R_k]$, $k = 1, \ldots, n-1$, form a basis for the image of the D-to-N map on $\mathcal{R}(b\Omega)$ as a mapping into $[\mathcal{V}]$. Indeed, if φ is a harmonic function with boundary values in $\mathcal{R}(b\Omega)$, we may subtract off a linear combination of the R_k to make the periods of the normal derivative of φ vanish, and then add them back to see that the normal derivative is in the same equivalence class as the linear combination of the R_k. Hence the $[R_k]$ span. They are linearly independent because the periods of the R_k are linearly independent.

We have now shown that the image of the D-to-N map on $\mathcal{R}(b\Omega)$ as a mapping into the $2(n-1)$ dimensional $[\mathcal{V}]$ is an $n-1$ dimensional subspace. It follows that the codimension of the image of the D-to-N map in $\mathcal{B}T + \overline{\mathcal{B}T}$ is $n-1$. We close this section by finding a more explicit description of these spaces.

First, we show that, for any point $a \in \Omega$, the function $K_a T$ is not the normal derivative of a harmonic function φ with rational boundary values $R(z, \bar{z})$ in $\mathcal{R}(b\Omega)$. (Here, we are using our notation $K_a(z) = K(z, a)$ as before.) Indeed, if it were, then

$$\varphi = R(z, S(z)) - \mathcal{G}$$

where $\mathcal{G}$ is a linear combination of functions G_w^m chosen to cancel the poles of $R(z, S(z))$. Arguing as in the derivation of equation (4.2) and (4.4), we see that the normal derivative of φ would be equal to

$$ih'T + \lambda T - \overline{\kappa T} \tag{6.3}$$

where $h(z) = R(z, S(z))$ and λ is in $\mathit{\Lambda}$ and has the same principal parts at the poles of h as $-h'$ and κ is in $\mathcal{B}$. The functions λ and κ are related via $\overline{\kappa T} = -\lambda T$ and the poles of λ cancel the poles of h'. If we apply this identity, we obtain

$$K_a = h' + 2\lambda$$

on the boundary. Since the functions involved are holomorphic, this identity extends to Ω. However, the poles of 2λ *do not* cancel the poles of h. Since K_a does not have poles, λ must be the zero function, and we see that $K_a = h'$ where h is holomorphic on Ω. We claim that this is not possible. Indeed, if $K_a = h'$, then pairing a complex polynomial $p(z)$ with K_a in the L^2 inner product on Ω yields

$$p(a) = \frac{1}{2i} \int_\Omega p \; \overline{h'} \; dz \wedge d\bar{z} = -\frac{1}{2i} \int_{b\Omega} p \; \overline{h} \; dz.$$

But h extends to the double as a meromorphic function. Hence there is a meromorphic function H on Ω such that $\overline{h} = H$ on the boundary, we now obtain that

$$p(a) = -\frac{1}{2i} \int_{b\Omega} p \, H \, dz$$

holds for all complex polynomials $p(z)$. The Residue Theorem now implies that H only has one pole on Ω, namely a simple pole at a, and so the meromorphic function on the double that represents the extension of h has only one simple pole on the double. For such a meromorphic function on the double of Ω to exist, Ω must be simply connected, a contradiction.

As before, let $\mathcal{B}_{n-1}$ denote the complex linear span of the functions K_{a_k}, $k = 1, \ldots, n-1$. If we were to apply the reasoning of the last paragraph to a nontrivial linear combination of the $K_{a_k} T$, we would be led to a meromorphic function on the double with exactly $n-1$ simple poles and no other poles. The Riemann-Roch

Theorem implies that no such meromorphic function exists and we conclude that this linear combination is not in $\mathcal{N}$. We have therefore shown that

$$[\mathcal{V}] = [\mathcal{N}] \oplus [\mathcal{B}_{n-1}T].$$

(Alternatively, since none of the $K_{a_k}T$ are in $\mathcal{N}$ and since the codimension of the span of these functions in $\mathcal{B}T + \overline{\mathcal{B}T}$ is $n-1$, we could arrive at the same conclusion.)

An interesting consequence of these deliberations is that, for each $j = 1, \ldots, n-1$, there are constants c_{jk} such that

$$\overline{K_{a_j}T} = h_j'T + \overline{H_j'T} + \sum_{k=1}^{n-1} c_{jk} K_{a_k} T \tag{6.4}$$

where h_j and H_j are holomorphic functions on Ω in $\mathcal{R}(\Omega)$. This observation gives rise to a method to determine if an element of $\mathcal{V}$ is in $\mathcal{N}$. Indeed, given an element $\kappa_1 T + \overline{\kappa_2 T}$ in $\mathcal{V}$, there are functions k_1 and k_2 in $\mathcal{B}_{n-1}$ such that $\kappa_j - k_j$ have vanishing periods, $j = 1, 2$. Now

$$\kappa_1 T + \overline{\kappa_2 T} = (\kappa_1 - k_1)T + \overline{(\kappa_2 - k_2)T} + k_1 T$$

plus $\overline{k_2 T}$, and we may use the equations (6.4) to convert the terms $\overline{k_2 T}$ to terms in $\mathcal{B}_{n-1}T$. Finally, when we combine these terms with $k_1 T$ in $\mathcal{B}_{n-1}T$, we see that our element is in $\mathcal{N}$ if and only if we get the zero function.

7. The Neumann problem

The Szegő projection can be used to solve the classical Neumann problem for the Laplacian in planar domains in much the same way that it was used above to solve the Dirichlet problem. This process is described in [3, p. 87]. On a double quadrature domain, both the Szegő kernel S_a and the Garabedian kernel L_a extend to the double (see [10]) and are therefore rational on the boundary. Also, the functions F_j' extend to the double (on area quadrature domains). If we combine these results with Theorem 20.1 in [3] and use the fact that the Szegő projection maps rational functions on the boundary to rational functions on the boundary (see [8]), we obtain the following result.

Theorem 7.1. *If ψ is a rational function on the boundary of a double quadrature domain Ω such that $\int_{b\Omega} \psi\, ds = 0$, then the solution to the Neumann problem with boundary data ψ is equal to*

$$h + \overline{H} + \sum_{k=0}^{n-1} c_k \omega_k$$

where h and H are holomorphic functions on Ω such that h' and H' extend meromorphically to the double (and are therefore rational on the boundary) and the c_k are constants.

8. Non-quadrature domains

All of the results of this paper carry over to non-quadrature domains if we define our basic objects differently. Suppose Ω is a bounded n-connected domain with C^∞ smooth boundary. In this context, let $\mathcal{R}(b\Omega)$ denote the space of C^∞ functions on the boundary that extend meromorphically to the double of Ω, let $\mathcal{R}(\Omega)$ denote the space of meromorphic functions on Ω that have boundary values in $\mathcal{R}(b\Omega)$, and let $\mathcal{R}'(\Omega)$ denote the space of functions that are derivatives of functions in $\mathcal{R}(\Omega)$. It is proved in [4] that there are two Ahlfors maps f_1 and f_2 associated to two (rather generic) points in Ω such that the meromorphic extensions to the double of Ω form a primitive pair for the double. Hence, the function spaces just described can all be expressed in terms of rational functions of f_1 and f_2. Since $\overline{f_j} = 1/f_j$ on the boundary $j = 1, 2$, these functions conveniently replace z and the Schwarz function in many situations.

The main theorems of the paper in this context can be stated as follows.

Theorem 8.1. *Given a point a in a bounded smooth finitely connected domain Ω,*

$$S_a\mathcal{R}(b\Omega) = \mathcal{S} + \mathcal{L},$$

where the function spaces on the right are understood to be restricted to the boundary. Furthermore,

$$S_a\mathcal{R}(\Omega) = \mathcal{S} + \mathcal{L}.$$

The Szegő projection maps $S_a\mathcal{R}(b\Omega)$ onto the Szegő span.

The Dirichlet problem can be solved for boundary data in $\mathcal{R}(b\Omega)$ by exactly the same methods we used in §3.

The theorem about the Bergman span also generalizes in a straightforward manner.

Theorem 8.2. *Suppose that Ω is a bounded smooth finitely connected domain. If Ω is simply connected, then*

$$\mathcal{R}'(\Omega) = \mathcal{B} + \Lambda.$$

On a multiply connected area quadrature domain, $\mathcal{R}'(\Omega)$ is comprised of the functions in $\mathcal{B}+\Lambda$ with vanishing periods. In both cases, the D-to-N map takes $\mathcal{R}(b\Omega)$ into $\mathcal{B}T + \overline{\mathcal{B}T}$. In case Ω is simply connected, this mapping is onto. Representations of functions ψ in $\mathcal{B}T + \overline{\mathcal{B}T}$ uniquely determine elements κ_1 and κ_2 in the Bergman span so that $\psi = \kappa_1 T + \overline{\kappa_2 T}$.

Finally, we remind the reader that we explained in §4 why Theorem 4.1 holds in general bounded smooth domains. We remark here that the general result could also be proved from scratch using the definitions in this section and by repeating the proof given in §4 using these definitions.

9. The solid Dirichlet problem on quadrature domains

Suppose that Ω is a simply connected area quadrature domain. Given a function v on Ω, we seek to find a solution φ to $\Delta\varphi = v$ with φ equal to zero on the boundary. The information we have gathered about the boundary Dirichlet problem on area quadrature domains yields results about the solid Dirichlet problem of a similar flavor. For example, we can deduce Ebenfelt's Theorem, that given a polynomial $v = P(z, \bar{z})$ in z and $\bar{z}$, the solution to the solid Dirichlet problem is algebraic. Indeed, we can antidifferentiate $P(z, \bar{z})$ in the z variable and the $\bar{z}$ variable to obtain a polynomial solution $q(z, \bar{z})$ to $\Delta q = 4\frac{\partial^2 q}{\partial z \partial \bar{z}} = v$, and then subtract off the solution to the boundary Dirichlet problem with boundary values given by q to deduce the result from the results of §3.

Thinking more deeply about this process, one is led to define the *real* Bergman span to be the set of all complex linear combinations of functions of the form $K_a^m(z)\overline{K_b^n(z)}$, i.e., the linear span of all functions of the form $\kappa_1\,\overline{\kappa_2}$ where $\kappa_j \in \mathcal{B}$, $j = 1, 2$. (I wish at this point that I had called the Bergman span the space of complex Bergman polynomials so that I could call the real Bergman span the space of real Bergman polynomials because these spaces share many properties of polynomials, but it is too late for that now.) Repeating the process above when v is in the real Bergman span and using Theorem 1.4 yields a solution to the solid Dirichlet problem which is a sum $u(z)$ of terms $h(z)\,\overline{H(z)}$ where h and H are holomorphic functions in $\mathcal{R}(\Omega)$. When we next subtract the solution to the boundary Dirichlet problem with boundary values given by the rational function $u(z)$ on the boundary, we see that the solution to the solid Dirichlet problem under such conditions is a rather concrete algebraic function.

As mentioned previously, it was shown in [7] that the Bergman span on an area quadrature domain contains the complex polynomials and all the complex rational functions that are holomorphic on Ω and with only residue free poles outside $\overline{\Omega}$. Hence the real Bergman span contains a rich variety of functions and is easily seen to be dense in $C^\infty(\overline{\Omega})$.

It is a rather easy exercise to show that the Bergman projection of an element of the real Bergman span is in the Bergman span. Indeed, if one pairs in L^2 an element $K_a^m\overline{K_b^n}$ with a function h in the Bergman space, one gets hK_b^n paired with K_a^m, which yields a fixed finite linear combination of derivatives of h at a. There is a function in the Bergman span that has the same effect on h when paired with h, and so we conclude that the element in the real Bergman span projects to this element in the Bergman span. Hence, we may now state the following theorem.

Theorem 9.1. *The Bergman projection associated to an area quadrature domain maps real polynomials to functions in the Bergman span (which are algebraic functions whose boundary values are given by rational functions).*

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Mathematics Department, Purdue University, West Lafayette, IN 47907

E-mail address: `bell@math.purdue.edu`