# SOMETHING ABOUT POISSON AND DIRICHLET

STEVEN R. BELL AND LUIS REYNA DE LA TORRE

ABSTRACT. We solve the Dirichlet problem in the unit disc and derive the Poisson formula using very elementary methods and explore consequent simplifications in other foundational areas of complex analysis.

## 1. MATHEMATICAL DNA

For reasons that have always been mysterious to the first author, he has often found himself thinking about the Dirichlet problem in the plane and has fought urges to look for explicit formulas for the Poisson kernels associated to various kinds of multiply connected domains, especially quadrature domains. He has obsessed about solutions to the Dirichlet problem with rational boundary data (see [4]) and he cannot stop thinking about the Khavinson-Shapiro conjecture about the same problem with polynomial data. His publication list is interspersed with papers where he has given in and found formulas for the Poisson kernel in terms of the Szegő kernel and in terms of Ahlfors maps. (See [2, 3] for an expository treatment of some of these results.) After thinking about the Dirichlet problem his entire adult life, he can solve it a different way every day of the week. He recently looked up his mathematical lineage at the Math Genealogy Project and found a possible explanation for his obsession. He is a direct mathematical descendent of both Poisson and Dirichlet. These problems are in his blood!

The authors worked together on a summer research project at Purdue University in 2018 to find a particularly elegant and simple way to approach these problems. We want to demonstrate here how Poisson and Dirichlet might solve their famous problems today if they had lived another 200 years and developed a major lazy streak. We assume that our reader has seen a traditional approach to this subject in a course on complex analysis and so will appreciate the novelty and smooth sailing of the line of reasoning here, but just in case the reader hasn't, we have tried to present the material in a way that can be understood assuming only a background in basic analysis.

We would like to thank Harold Boas for reading an early draft of this work and making many valuable suggestions for improvement. Harold knows a lot about the family business because he, like the first author, was a student of Norberto Kerzman at MIT in the late 1970's.

## 2. Harmonic functions

We define a harmonic function on a domain $\Omega$ in the complex plane to be a continuous complex valued function $u(z)$ on the domain that satisfies the *averaging property* on $\Omega$, meaning that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta$$

whenever $D_r(a)$, the disc of radius $r$ about $a$, is compactly contained in $\Omega$. It is well known that this definition of harmonic function is equivalent to all the other standard definitions (see, for example, Rudin [7, Chap. 11]), and we will demonstrate this in very short order. Since we will explore other definitions of harmonic, we will emphasize that we are currently thinking of harmonic functions as being defined in terms of an averaging property by calling them harmonic-ave functions.

We first note that analytic polynomials are harmonic-ave. Indeed, if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, the averaging property is clear on discs centered at the origin because the constant function $a_0$ obviously satisfies the averaging property at the origin and so does $z^n$ for $n \geq 1$ because

$$\int_0^{2\pi} (re^{i\theta})^n \, d\theta = r^n \int_0^{2\pi} \cos(n\theta) \, d\theta + ir^n \int_0^{2\pi} \sin(n\theta) \, d\theta = 0 = 0^n.$$

To see that the averaging property holds at a point $a \in \mathbb{C}$, write

$$P(z) = P((z - a) + a)$$

and expand to get a polynomial in $(z - a)$. Now the argument we used at the origin can be applied to discs centered at $a$.

It follows that the averaging property also holds for conjugates of polynomials in $z$, so polynomials in $\bar{z}$ are harmonic-ave. Note also that complex valued functions are harmonic-ave if and only if their real and imaginary parts are both harmonic-ave.

The *Dirichlet problem* on the unit disc is: given a continuous real valued function $\varphi$ on the unit circle, find a real valued function $u$ that is continuous on the closure of the unit disc with boundary values given by $\varphi$ such that $u$ is harmonic on $D_1(0)$. We will find a solution to this problem that is harmonic in the averaging sense in the next section. To do so, we will need to know the elementary fact that a continuous real valued function on the unit circle can be uniformly approximated on the unit circle by a real polynomial $p(x, y)$. We will now demonstrate this little fact, assuming the Weierstrass theorem about the density of real polynomials of one variable among continuous functions on closed subintervals of the real line.

Suppose we are given a continuous real valued function $\varphi$ on the unit circle. We wish to find a real polynomial $p(x, y)$ that is uniformly close to $\varphi$ on the unit circle. Note that we may assume that $\varphi(\pm 1) = 0$ because we may subtract

a polynomial function of the form $ax + b$ to make $\varphi$ zero at $\pm 1$. Define two continuous functions on $[-1, 1]$ via

$$h_{\text{top}}(x) = \varphi(x + i\sqrt{1 - x^2})$$

and

$$h_{\text{bot}}(x) = \varphi(x - i\sqrt{1 - x^2}).$$

We can uniformly approximate $h_{\text{top}}$ and $h_{\text{bot}}$ on $[-1, 1]$ by real polynomials $p_{\text{top}}(x)$ and $p_{\text{bot}}(x)$. Next, let $\chi_\epsilon(y)$ be a continuous function that is equal to one for $y > \epsilon$, equal to zero for $y < -\epsilon$ and follows the line connecting $(-\epsilon, 0)$ to $(\epsilon, 1)$ for $-\epsilon \le y \le \epsilon$. Let $p_\epsilon(y)$ be a polynomial in $y$ that is uniformly close to $\chi_\epsilon$ on $[-1, 1]$. Now, because $\varphi$ vanishes at $\pm 1$, the polynomial

$$p_{\text{top}}(x)p_\epsilon(y) + p_{\text{bot}}(x)p_\epsilon(-y)$$

approximates $\varphi$ on the unit circle, and the approximation can be improved uniformly by shrinking $\epsilon$ and improving the approximations of the other functions involved.

We are now in position to solve the Dirichlet problem on the disc, but before we begin in earnest, this is a good place to emphasize that Green's theorem for the unit disc depends on nothing more than the fundamental theorem of calculus from freshman calculus. Indeed, let $P(x, y)$ be a $C^1$-smooth function. Let $C_{\text{top}}$ denote the top half of the unit circle parameterized in the clockwise sense by $z(x) = x + iy_{\text{top}}(x)$, $-1 \le x \le 1$, where $y_{\text{top}}(x) = \sqrt{1 - x^2}$, and let $C_{\text{bot}}$ denote the bottom half of the unit circle parameterized in the counterclockwise sense by $z(x) = x + iy_{\text{bot}}(x)$, $-1 \le x \le 1$, where $y_{\text{bot}}(x) = -\sqrt{1 - x^2}$. Note that the unit circle $C_1(0)$ parametrized in the counterclockwise sense is given by $C_{\text{bot}}$ followed by $-C_{\text{top}}$. Drum roll...

$$\int_{C_1(0)} P\, dx = \left( \int_{C_{\text{bot}}} - \int_{C_{\text{top}}} \right) P\, dx = \int_{-1}^{1} [P(x, y_{\text{bot}}(x)) - P(x, y_{\text{top}}(x))]\ dx$$

$$= -\int_{-1}^{1} \left( \int_{y_{\text{bot}}(x)}^{y_{\text{top}}(x)} \frac{\partial P}{\partial y}(x, y)\ dy \right) dx = \iint_{D_1(0)} -\frac{\partial P}{\partial y}\ dx \wedge dy.$$

The other half of Green's formula follows by repeating the argument using the words left and right in place of top and bottom. This argument on the disc can be easily generalized to demonstrate Green's theorem on any region that can be cut up into regions that have a top boundary curve and a bottom curve and a left curve and a right curve. An annulus centered at the origin cut into four regions by the two coordinate axes is such a domain. We will need Green's theorem later in the paper when we study analytic functions from a philosophical point of view inspired by our observations about harmonic functions and the Dirichlet problem.

### 3. Solution of the Dirichlet problem on the unit disc

The unit disc has the special feature that, given polynomial data $\varphi$, it is straightforward to write down a polynomial solution to the Dirichlet problem with boundary values given by $\varphi$. Indeed, a polynomial $p(x, y)$ in the real variables $x$ and $y$ can be converted to a polynomial in $z$ and $\bar{z}$ by replacing $x$ by $(z + \bar{z})/2$ and $y$ by $(z - \bar{z})/(2i)$ and expanding. It is now an easy matter to extend the individual terms in the sum to harmonic-ave functions on the disc by noting that

$$z^n \bar{z}^m$$

is equal to one on the unit circle if $n = m$, equal to $z^{n-m}$ on the circle if $n > m$, and equal to $\bar{z}^{m-n}$ on the circle if $m > n$, each of which is harmonic-ave inside the unit circle.

Now, given a continuous real valued function $\varphi$ on the unit circle, there is a sequence of real valued polynomials $p_n(x, y)$ that converges uniformly to $\varphi$ on the unit circle. Let $u_n$ be the polynomial harmonic-ave extension of $p_n$ to the disc described in the paragraph above. Note that $u_n$ can be expressed as a constant plus a polynomial in $z$ that vanishes at the origin plus a polynomial in $\bar{z}$ that also vanishes at the origin. We now claim that the functions $u_n$ converge uniformly on the closed disc to a solution of the Dirichlet problem. To see this, we must first show that real valued harmonic-ave functions $u$ on the disc that extend continuously to the closure satisfy the *maximum principle* in the form

$$\max\{u(z) : |z| \leq 1\} = \max\{u(e^{i\theta}) : 0 \leq \theta \leq 2\pi\}.$$

Indeed, if the maximum value of such a function $u$ occurs at a point $z_0$ inside the unit circle, we can express the value of $u$ at $z_0$ as an average of $u$ over a small circle centered at $z_0$. We can let the radius of that circle increase until the circle touches the unit circle at a single point. The averaging property holds on the limiting circle because of uniform continuity. Let $M$ denote the maximum value $u(z_0)$. Now, in order for the average of a continuous function that is less than or equal to $M$ over that circle to be equal to $M$, it must be that $u$ is equal to $M$ on the whole circle. Hence the value of $u$ at the point where the inner circle touches the unit disc must also be $M$. This proves the maximum principle inequality for real harmonic-ave functions. The minimum principle follows by applying the maximum principle to $-u$.

Because the $p_n$ converge to $\varphi$ uniformly on the unit circle, the sequence $\{p_n\}$ is uniformly Cauchy on the unit circle, i.e., given $\epsilon > 0$, there is an $N$ such that $|p_n - p_m| < \epsilon$ on the unit circle when $n$ and $m$ are greater than $N$. The maximum and minimum principle inequalities applied to the imaginary parts of $u_n$ (which are zero on the unit circle) allow us to conclude that the functions $u_n$ are *real* valued. Furthermore, the maximum and minimum principles applied to $u_n - u_m$ show that the uniformly Cauchy estimates for the sequence $\{p_n\}$ on the unit circle extend to hold for the sequence $\{u_n\}$ on the whole closed unit disc, showing that $\{u_n\}$ is uniformly Cauchy on the closed unit disc. Hence, since the $u_n$ are continuous, they converge uniformly on the closed unit disc to a

continuous function $u$ that is equal to $\varphi$ on the unit circle. Finally, it is clear that $u$ is harmonic-ave on the inside of the unit circle because the averaging property is preserved under uniform limits. We have solved the Dirichlet problem in a purely existential manner without ever differentiating a function! We now turn to Poisson's problem of finding a *formula* for our solution.

## 4. Poisson's formula

Notice that the set of harmonic-ave functions

$$\{1, z^n, \bar{z}^n : n = 1, 2, 3, \dots\}$$

is orthonormal under the inner product

$$\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, \overline{v(e^{i\theta})} \, d\theta.$$

Define $K_N(z, w)$ via

$$K_N(z, w) := 1 + \sum_{n=1}^N z^n \bar{w}^n + \sum_{n=1}^N \bar{z}^n w^n,$$

and observe that if $u(z) = z^n$, then

(4.1) $$u(z) = \frac{1}{2\pi} \int_0^{2\pi} K_N(z, e^{i\theta}) \, u(e^{i\theta}) \, d\theta$$

for $z \in D_1(0)$ if $N \geq n$ because of the orthonormality of the terms in the sum. The same is true if $u(z) \equiv 1$ or $u(z) = \bar{z}^n$. We conclude that, if $u$ is the solution of the Dirichlet problem for polynomial data $p$ of degree $n$ as constructed in the previous section, then formula (4.1) holds for $u(z)$ for $z \in D_1(0)$ when $N > n$.

Using the famous geometric series formula,

$$1 + \zeta + \cdots + \zeta^N = \frac{1}{1 - \zeta} - \frac{\zeta^{N+1}}{1 - \zeta},$$

we see that

$$K_N(z, w) = 1 + (z\,\bar{w}) \sum_{n=0}^{N-1} z^n \bar{w}^n + (\bar{z}\,w) \sum_{n=0}^{N-1} \bar{z}^n w^n$$

converges uniformly in $w = e^{i\theta}$ when $z \in D_1(0)$ to

$$K(z, w) := 1 + \frac{z\,\bar{w}}{1 - z\,\bar{w}} + \frac{\bar{z}\,w}{1 - \bar{z}\,w},$$

and the error $\mathcal{E}_N = |K_N - K|$ is controlled via

$$\mathcal{E}_N(z, w) \leq \frac{2|z|^{N+1}}{1 - |z|}$$

when $z \in D_1(0)$ and $|w| = 1$. Hence, it follows by taking uniform limits that formula (4.1) holds for $z \in D_1(0)$ with the $N$ removed for the polynomials $u_n$ that we constructed from polynomial boundary data $p_n$. We can now let

the polynomials $p_n$ tend uniformly to $\varphi$ and use the fact proved above that the corresponding solutions $u_n$ to the Dirichlet problem with boundary data $p_n$ converge uniformly to a solution $u$ of the Dirichlet problem to obtain Poisson's famous formula for the solution to the Dirichlet problem,

$$(4.2) \qquad u(z) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{i\theta}) \varphi(e^{i\theta}) \, d\theta.$$

This formula reveals that the solution $u$ can be written

$$u(z) = a_0 + h(z) + \overline{h(z)}$$

where $a_0$ is the (real valued) average of $\varphi$ on the unit circle and $h$ is an analytic function on $D_1(0)$ that vanishes at the origin given via

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{ze^{-i\theta}}{1 - ze^{-i\theta}} \, \varphi(e^{i\theta}) \, d\theta = \frac{z}{2\pi i} \int_{C_1} \frac{\varphi(w)}{w(w - z)} \, dw,$$

where $C_1$ denotes the unit circle parameterized in the standard sense using $w = e^{i\theta}$ and $dw = ie^{i\theta} \, d\theta$. It is now a rather easy exercise to take limits of complex difference quotients to see that complex derivatives in $z$ can be taken under the integral sign in the definition of $h$. Hence, $h(z)$ is infinitely complex differentiable. It follows from the Cauchy-Riemann equations that $u$ is a $C^\infty$-smooth real valued function on $D_1(0)$ in $x$ and $y$ that satisfies the Laplace equation there and solves the Dirichlet problem. Furthermore, $u$ is the real part of an infinitely complex differentiable function $H = a_0 + h/2$ on $D_1(0)$.

The maximum principle yields that $u$ is the *unique* solution to the problem in the realm of harmonic functions understood in the sense of averaging. We will see in the next section that it is also the unique solution among harmonic functions defined in the traditional sense of satisfying the Laplace equation.

We remark here that rather simple algebra reveals the well-known formulas for the Poisson kernel,

$$K(z, w) = \operatorname{Re} \frac{1 + z\,\bar{w}}{1 - z\,\bar{w}} = \operatorname{Re} \frac{w + z}{w - z}$$

and

$$K(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}.$$

It is a routine matter to extend the above line of reasoning to any disc (either by repeating the argument or making a complex linear change of variables $Az + B$). Since a complex valued function is harmonic-ave if and only if its real and imaginary parts are harmonic-ave, it follows from our work that a complex valued harmonic-ave function is given locally by $g + \overline{G}$ where $g$ and $G$ are infinitely complex differentiable functions.

At this point, it would be tempting to experiment with thinking of analytic functions as being harmonic-ave functions that do not involve the antianalytic $\overline{G}$ parts. The formula for $h$ above reveals that the analytic $g$ part is locally a uniform

limit of analytic polynomials. Since $\{1, z^n : n = 1, 2, 3, \ldots\}$ are orthonormal on the unit circle, we could let

$$k_N(z, w) = 1 + \sum_{n=1}^{N} z^n \bar{w}^n$$

and use the same line of reasoning that we used earlier in this section to conclude that

$$(4.3) \qquad f(z) = \frac{1}{2\pi} \int_0^{2\pi} k_N(z, e^{i\theta}) f(e^{i\theta}) \, d\theta$$

if $f(z)$ is equal to 1 or $z^n$ with $n \leq N$. The geometric series estimate we used above shows that $k_N(z, w)$ converges uniformly in $w$ on the unit circle for fixed $z$ in $D_1(0)$ to

$$k(z, w) := \frac{1}{1 - z\,\bar{w}}.$$

Hence, we can let $N \to \infty$ in (4.3) to see that

$$(4.4) \qquad f(z) = \frac{1}{2\pi} \int_0^{2\pi} k(z, e^{i\theta}) f(e^{i\theta}) \, d\theta$$

when $f$ is an analytic polynomial. Finally, if $f$ is a uniform limit of analytic polynomials on an open set containing the closed unit disc, we may conclude that $f$ satisfies identity (4.4), too. The identity can easily be seen to be the classical Cauchy integral formula on the unit disc, and from this point, the theory of analytic functions would gush forth. In particular, analytic functions would be seen to be infinitely complex differentiable and given locally by convergent power series. We will explore this idea and various other alternate ways of thinking about analytic functions after we verify that our definition of harmonic functions via an averaging property gives rise to the same set of functions as any of the more standard definitions.

## 5. Traditional definitions of harmonic functions

Some complex analysis books define harmonic functions to be twice continuously differentiable functions that satisfy the Laplace equation. With this definition, one can use Green's identities on an annulus (which we pointed out in section 1 to be quite elementary) to show that such harmonic functions satisfy the averaging property. Hence, this class of functions could be seen to be the same as harmonic-ave functions. We won't pursue this idea here because we can easily prove something stronger with less effort.

It is most gratifying to define harmonic functions to be merely continuous functions $u$ whose first partial derivatives exist and whose second partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist and satisfy the Laplace equation. Call such functions harmonic-pde. We will now adapt a classic argument to show that the class of harmonic-pde functions agrees with our class of continuous functions that satisfy the averaging property on an open set. Indeed, if $u$ is a harmonic-pde

function defined on an open set containing the closed unit disc, then $u$ minus the harmonic-ave function $U$ that we constructed solving the Dirichlet problem on the unit disc with the same boundary values as $u$ on the unit circle, if not the zero function, would have either a positive maximum or a negative minimum. Suppose it has a positive maximum $M > 0$ at a point $z_0$ in $D_1(0)$. Choose $\epsilon$ with $0 < \epsilon < M$. Now

$$v(z) := u(z) - U(z) + \epsilon|z|^2$$

is equal to $\epsilon$ on the unit circle and attains a positive value at $z_0$ that is larger than $\epsilon$. Hence, $v$ attains a positive maximum at some point $w_0$ in $D_1(0)$. The Laplacian of $v$ at $w_0$ is equal to $4\epsilon$, which is strictly positive. However, the one variable second derivative test from freshman calculus applied to $v$ in the $x$-direction yields that $\partial^2 v/\partial x^2$ must be less than or equal to zero. (If it were positive, $v$ could not have a local maximum at $w_0$.) Similarly, the second derivative test in the $y$-direction yields that $\partial^2 v/\partial y^2$ must be less than or equal to zero. We conclude that the Laplacian of $v$ at $w_0$ must be less than or equal to zero, which is a contradiction because the Laplacian is strictly positive on $D_1(0)$. This shows that $u - U$ cannot have a positive value. If we replace $u - U$ by $U - u$, the same reasoning shows that $U - u$ cannot have a positive value. Hence $U - u \equiv 0$ on the unit disc and we have shown that $u$, like $U$, is harmonic in the sense of being averaging. Finally, because the operations of translating and scaling preserve harmonic functions in both senses of the word, we can translate and scale any disc to the unit disc to be able to deduce the equivalence of the two definitions of harmonic on any open set. We now stop hyphenating harmonic and turn to hyphenating analytic.

## 6. ANALYTIC FUNCTIONS

When the 200+ year old Poisson and Dirichlet took a break from harmonic functions, they might have turned their attention to similar considerations applied to analytic functions, which satisfy both the averaging property on discs and Cauchy's theorem for circles.

In order to understand the rest of this section, the reader will need to know (or look up) the definition of the complex contour integral $\int_\gamma f \, dz$ along a curve $\gamma$, the basic estimate

$$\left| \int_\gamma f \, dz \right| \leq \sup\{|f(z)| : z \in \gamma\} \cdot \text{Length}(\gamma),$$

and the fundamental theorem of calculus $\int_\gamma f' \, dz = f(b) - f(a)$ for complex contour integrals, assuming that $f$ is continuously complex differentiable and that $\gamma$ starts at $a$ and ends at $b$. We will also use the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

These two very important operators can be "discovered" by writing $dz = dx + i\, dy$ and $d\bar{z} = dx - i\, dy$ and manipulating

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

to appear in the form

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}.$$

The condition $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to writing the Cauchy-Riemann equations for the real and imaginary parts of $f$. If $h(z)$ is a complex differentiable function, then $\frac{\partial h}{\partial \bar{z}} = 0$ and $\frac{\partial h}{\partial z} = h'(z)$. Furthermore, $\frac{\partial \bar{h}}{\partial z} = 0$ and $\frac{\partial \bar{h}}{\partial \bar{z}} = \overline{h'(z)}$.

We will denote the boundary circle of the disc $D_r(a)$ parameterized in the counterclockwise sense by $C_r(a)$. Writing out the real and imaginary parts of the contour integral of a complex function $f$ around $C_r(a)$ and applying Green's theorem for a disc to the real and imaginary parts yields, what we like to call, the *complex Green's theorem* for a disc,

$$\int_{C_r(z_0)} f\, dz = 2i \iint_{D_r(z_0)} \frac{\partial f}{\partial \bar{z}}\, dA,$$

where $dA$ denotes the element of area $dx \wedge dy$. (Using the real Green's theorem to prove the complex Green's theorem is quick and easy, but a more civilized way to deduce the theorem would be to note that

$$d(f\, dz) = \frac{df}{dz}\, dz \wedge dz + \frac{df}{d\bar{z}}\, d\bar{z} \wedge dz = 0 + \frac{df}{dz}\, (2i\, dx \wedge dy)$$

and to apply Stokes' theorem.)

Define a continuous complex valued function $f$ to be analytic-circ on a domain $\Omega$ if it is harmonic on $\Omega$ (and so satisfies the averaging property) and, for each closed subdisc of $\Omega$, the complex contour integral of $f$ over the boundary circle is zero.

Complex polynomials are easily seen to be analytic-circ because the monomials are complex derivatives of monomials one degree higher and the fundamental theorem of calculus for complex contour integrals reveals that the integrals around closed curves are zero. It now follows that functions that are the uniform limit of complex polynomials (in $z$) on each closed subdisc of $\Omega$ are analytic. We will now show that functions that are analytic-circ must be the uniform limit of complex polynomials on each closed subdisc of $\Omega$.

Suppose $f$ is analytic-circ on $\Omega$. Let $D_r(z_0)$ be a disc that is compactly contained in $\Omega$ and let $C_r(z_0)$ denote the boundary circle parameterized in the counterclockwise sense. The complex Green's theorem for a disc yields that

$$0 = \int_{C_r(z_0)} f\, dz = 2i \iint_{D_r(z_0)} \frac{\partial f}{\partial \bar{z}}\, dA,$$

where $dA$ denotes the element of area $dx \wedge dy$. Since this integral is zero for every disc compactly contained in $\Omega$, it follows that $\partial f / \partial \bar{z} \equiv 0$ on $\Omega$, i.e., that

the real and imaginary parts of $f$ satisfy the Cauchy-Riemann equations. Since harmonic functions are infinitely differentiable, the textbook proof that $f$ is complex differentiable can now be applied.

Since being analytic-circ is a local property and is invariant under changes of variables of the form $Az + B$, we may restrict our attention to a function $f$ that is analytic-circ on a neighborhood of the closed unit disc. We can gain more insight into the implications of the definition by noting that such a harmonic function is given by $g + \overline{G}$ where $g$ and $G$ are infinitely complex differentiable. It follows from our complex Green's calculation above that $\partial f / \partial \bar{z} = \overline{G'} \equiv 0$ on $\Omega$, and that $f$ is therefore equal to the analytic part $g$ plus a constant. From this point, it follows that $f$ is given by a Cauchy integral formula and we merge into the fast lane of the classical theory of analytic functions.

Of course, the traditional way to define analytic functions is as complex differentiable functions on open sets. Call such functions analytic-diff. We will now show that the traditional definition leads to the same class of functions that we have defined as being analytic-circ.

Suppose that $f$ is an analytic-diff function defined on an open set containing the closed unit square $\mathcal{S} := [0, 1] \times [0, 1]$. We will show that the complex integral of $f$ around the counterclockwise perimeter curve $\sigma$ of the square must be zero. The well known argument will be a beautiful bisection method tracing back to Goursat.

It follows from our assumption that $f$ can be locally well approximated by a complex linear function in the following sense. Suppose $a$ is a point in $\mathcal{S}$, and let $\epsilon > 0$ be given. Since $f'(a)$ exists, there is a $\delta > 0$ such that

$$\frac{f(z) - f(a)}{z - a} = f'(a) + E_a(z)$$

where $|E_a(z)| < \epsilon$ when $|z - a| < \delta$, $z \neq a$. Define $E_a(a)$ to be zero to make $E_a$ continuous at $a$ and so as to be able to assert that

$$f(z) = f(a) + f'(a)(z - a) + E_a(z)(z - a)$$

on the whole open set where $f$ is defined and $E_a$ is a continuous function in $z$ on that set. Furthermore, $|E_a(z)| < \epsilon$ on $D_\delta(a)$. The complex integral of the polynomial $f(a) + f'(a)(z - a)$ around any square is zero because first degree polynomials are derivatives of second degree polynomials. If $\sigma_h$ is the counterclockwise boundary curve of a small square $\mathcal{S}_h$ of side $h$ contained in $D_\delta(a)$ that contains the point $a$, it follows that

$$\left| \int_{\sigma_h} f(z) \, dz \right| = \left| \int_{\sigma_h} E_a(z)(z - a) \, dz \right| \leq \epsilon(\sqrt{2}h)(4h).$$

Hence,

$$(6.1) \qquad\qquad \left| \int_{\sigma_h} f \, dz \right| \leq (4\sqrt{2})\epsilon \, \text{Area}\,(\mathcal{S}_h).$$

We will now follow a version of Goursat's famous argument to explain how this could be made *too small* if $\int_\sigma f \; dz$ were not zero.

Indeed, suppose that $I := \int_\sigma f \; dz$ is not equal to zero. Note that $I$ is equal to the sum of the integrals around the four counterclockwise squares obtained by cutting the big square into four equal squares of side $1/2$ since the integrals along the common edges cancel. For these four integrals to add up to the non-zero value $I$, the modulus of at least one of them must be greater than or equal to $|I|/4$. Name such a square $\mathcal{S}_1$ and its counterclockwise boundary curve $\sigma_1$. Note that

$$\left| \int_{\sigma_1} f \; dz \right| \geq |I| \cdot \mathrm{Area}\,(\mathcal{S}_1).$$

We may now dice up $\mathcal{S}_1$ into four equal subsquares and repeat the argument to obtain a square $\mathcal{S}_2$ with boundary curve $\sigma_2$ such that the modulus of the integral of $f$ around $\sigma_2$ is greater than or equal to $|I|$ times the area of $\mathcal{S}_2$. Continuing in this manner, we obtain a nested sequence of closed squares $\{\mathcal{S}_n\}_{n=1}^\infty$ with boundary curves $\sigma_n$, the diameters of which tend to zero as $n \to \infty$, such that

$$(6.2) \qquad\qquad \left| \int_{\sigma_n} f \; dz \right| \geq |I| \cdot \mathrm{Area}\,(\mathcal{S}_n).$$

There is a unique point $a$ that belongs to all the squares. Now, given an $\epsilon$ less than $|I|/(4\sqrt{2})$, the squares that eventually fall in the disc $D_\delta(a)$ that we specified above satisfy both area inequalities (6.1) and (6.2), which are incompatible. This contradiction shows that $I$ must be zero!

Since any square can be mapped to the unit square via a mapping of the form $Az + B$, it follows from a simple change of variables that the integral of an analytic-diff function around any square must be zero. Furthermore, any rectangle can be approximated by a rectangle subdivided into a union of $n \times m$ squares. It follows that the integral of an analytic-diff function around any rectangle must be zero.

From this point, there are several standard arguments to prove the Cauchy integral formula on a disc for such functions (see Ahlfors [1, p. 109] or Stein [9, p. 37]). It then follows from the Cauchy integral formula that such a function would be locally the uniform limit of analytic polynomials, and so the function would be analytic-circ. However, we can simplify these standard arguments by using some of the power of our work on harmonic functions in the previous sections.

Suppose that $f$ is analytic-diff on an open disc. Since $f$ is complex differentiable, it is continuous. Define $F(z)$ at a point $z$ in the disc by the integral of $f$ along a horizontal "zig" from the center followed by a vertical "zag" connecting to the point $z$. The fundamental theorem of calculus reveals that

$$\frac{\partial}{\partial y} F(x + iy) = i f(x + iy).$$

Since the integral of $f$ around rectangles is zero, we could also define $F$ via an integral along a vertical zag followed by a horizontal zig. Using this definition, the fundamental theorem of calculus shows that

$$\frac{\partial}{\partial x}F(x+iy) = f(x+iy).$$

Hence, $F$ is a continuously differentiable function whose real and imaginary parts satisfy the Cauchy-Riemann equations, and furthermore, is a complex differentiable function such that $F' = f$ on the disc. Repeat this construction to get a twice continuously complex differentiable function $G$ such that $G'' = f$.

Now, since $G$ is twice continuously complex differentiable, it is easy to use the Cauchy-Riemann equations to show that the real and imaginary parts of $G$ are harmonic functions. Indeed, if $G(x+iy) = u(x,y) + iv(x,y)$, then the Cauchy-Riemann equations yield that

$$G' = u_x + iv_x = v_y - iu_y$$

and

$$G'' = u_{xx} + iv_{xx} = -u_{yy} - iv_{yy}$$

and we see that $u$ and $v$ satisfy the Laplace equation by equating the real and imaginary parts of $G''$. Our work in previous sections shows that these harmonic functions are $C^\infty$-smooth and it follows that the real and imaginary parts of $f$ are $C^\infty$-smooth and satisfy the Laplace equation and the Cauchy-Riemann equations. We may now use Green's theorem on a disc to prove the Cauchy theorem for $f$ on discs. Hence $f$ is analytic-circ and we have proved the equivalence of the definitions, revealing that $f$ is locally the uniform limit of complex polynomials and is given by the Cauchy integral formula. The shortcuts we have revealed in the theory of analytic functions deliver us to page 114 of Ahlfors.

Before we conclude this section, we present one last alternate way to define analytic functions that might be of interest to experienced analysts. The result is known (see for example, Springer [8] or Globevnik [5]), but we are in a position to prove it rather efficiently here. We now will define a continuous complex valued function to be analytic-ave on the unit disc if $f(z)(z-a)$ satisfies the averaging property on circles $C_r(a)$ contained in the disc, i.e., if

$$0 = \int_0^{2\pi} f(a+re^{i\theta})(re^{i\theta})\,d\theta$$

whenever the closure of $D_r(a)$ is contained in the unit disc. Note that this condition is equivalent to the condition that

$$0 = \int_{C_r(a)} f\,dz$$

for each such circle. We will now prove that analytic-ave functions are analytic in the usual sense. This result can be viewed as a version of Morera's theorem saying that a continuous complex valued function that satisfies the Cauchy theorem on circles must be analytic.

Let $\chi(t)$ be an real valued non-negative function in $C^\infty[0,1]$ that is equal to one for $t < \frac{1}{2}$ and equal to zero for $t > \frac{3}{4}$. Define

$$\varphi(z) = c\chi(|z|^2)$$

where $c$ is chosen so that $\int \varphi \, dA = 1$. Let $\varphi_\epsilon$ denote the approximation to the identity given by

$$\varphi_\epsilon(z) = \frac{1}{\epsilon^2}\varphi(z/\epsilon).$$

The proof of our claim rests on a straightforward calculation that shows that

$$\frac{\partial}{\partial\bar{z}}\varphi_\epsilon(z-w)$$

is $(z-w)$ times a function $\psi_\epsilon(z-w)$ that is radially symmetric about $z$ in $w$. In fact,

$$\psi_\epsilon(z) = \frac{c}{\epsilon^4}\chi'(|z|^2/\epsilon^2).$$

The calculation hinges on the chain rule plus the fact that

$$\frac{\partial}{\partial\bar{z}}|z^2| = \frac{\partial}{\partial\bar{z}}(z\,\bar{z}) = z.$$

Given a continuous function $f$ on the unit disc such that $f(z)(z-a)$ satisfies the averaging property on circles $C_r(a)$ compactly contained in the disc, let $f_\epsilon = \varphi_\epsilon * f$ for small $\epsilon > 0$. Note that $f_\epsilon$ is $C^\infty$ smooth on $D_{1-\epsilon}(0)$ and that $f_\epsilon$ converges uniformly on compact subsets of the unit disc to $f$ as $\epsilon \to 0$. One can differentiate under the integral in the convolution formula to see that

$$\frac{\partial f_\epsilon}{\partial\bar{z}} = \frac{\partial\varphi_\epsilon}{\partial\bar{z}} * f = \iint_{w\in D_1(0)} \frac{\partial\varphi_\epsilon}{\partial\bar{z}}(z-w)\,f(w)\,dA,$$

and the observation about the radially symmetric function and our hypothesis about $f$ allows us write the integral in polar coordinates about $z$ to conclude that $f_\epsilon$ satisfies the Cauchy-Riemann equations, and so is analytic-diff on $D_{1-\epsilon}(0)$, and consequently, is analytic-circ there, too. It is easy to see that uniform limits of analytic-circ functions are analytic-circ. We conclude that $f$ is analytic-circ and so analytic in any sense of the word.

## 7. The Dirichlet problem in more general domains

We solved the Dirichlet problem on the unit disc, given polynomial boundary data, by explicitly extending individual terms $z^n\bar{z}^m$ as harmonic polynomials. Another way to approach this problem is via linear algebra. Suppose a domain $\Omega$ is described via a real polynomial defining function $r(x,y)$ (meaning that $\Omega = \{x + iy : r(x,y) < 0\}$), where $r(x,y)$ is of degree two. For $\Omega$ to be a bounded domain, it is clear that the boundary of $\Omega$, which is the zero set of $r$, must be a circle or an ellipse. Let $\Delta$ denote the Laplace operator. Now, the map $\mathcal{F}$ that takes a polynomial $p(x,y)$ to the polynomial $\Delta(rp)$ maps the finite dimensional vector space $\mathcal{P}_N$ of polynomials of degree $N$ or less to itself. (Multiplying by $r$ increases the degree by two, and applying the second order

operator $\Delta$ brings it back down by two.) We claim that the map $\mathcal{F}$ is one-to-one on $\mathcal{P}_N$, and therefore *onto*. Indeed, if $\Delta(rp)$ is the zero polynomial, then $rp$ is a harmonic polynomial that vanishes on the boundary. The maximum principle implies that it must be the zero polynomial. Consequently, $p$ must be the zero polynomial, and this proves that $\mathcal{F}$ is a one-to-one linear mapping of a finite dimensional vector space into itself, and so also onto. Now, to solve the Dirichlet problem on $\Omega$, given polynomial boundary data $q(x, y)$, we know there is polynomial $p$ such that $\Delta(rp) = \Delta q$. The polynomial $q - rp$ is harmonic on $\Omega$ and equal to $q$ on the boundary. It solves the Dirichlet problem. We could have solved the Dirichlet problem on the unit disc in the realm of polynomials without ever writing a formula down! Now we can solve the Dirichlet problem on an ellipse using the same procedure that we did on the disc. (However, the next obvious step, to try to write down a Poisson integral formula on the ellipse, gets more complicated because the monomials are not orthonormal in the boundary inner product of the ellipse.)

The Khavinson-Shapiro conjecture states that discs and ellipses are the only domains in the plane having the property that solutions to the Dirichlet problem with polynomial data must be polynomials. It is tantalizing that it seems so much harder to settle this question than the same problem with the word "polynomial" replaced by "rational." Only discs have the property that solutions to the Dirichlet problem with rational boundary data must be rational (see [4]).

For the remainder of this section, we will let our Poisson and Dirichlet urges run rampant and explain how the ideas in the previous sections might be used to solve the Dirichlet problem on more general domains. We will dispense with proofs and follow a line of bold declarations. The interested reader can find a more sober exposition of some of these ideas in Chapters 22 and 34 of [2] and in [3].

To solve the Dirichlet problem on a domain bounded by a Jordan curve, one can use Carathéodory's theorem (the theorem that states that the Riemann map associated to such a domain extends continuously to the boundary and maps the boundary one-to-one onto the unit circle) to be able to pull back solutions to the problem on the unit disc to the domain. But we wonder if there might be a more elementary way to do it.

Gustafsson's theorem [6] states that a bounded finitely connected domain with $n$ continuous simple closed boundary curves can be mapped to an $n$-connected *quadrature domain* $\Omega$ with smooth real analytic boundary via a conformal mapping that is continuous up to the boundary and as close to the identity map in the uniform topology of the closure of the domain as desired. Such a "nearby" quadrature domain has the property that the average of an analytic function over the domain with respect to area measure is a finite linear combination of values of the function and its derivatives at finitely many points in the domain. The resulting "quadrature identity" is the same for all analytic functions that are square-integrable with respect to area measure on the domain. Smooth real analytic curves have "Schwarz functions" $S(z)$ that are analytic on a neighborhood

of the curve and satisfy $S(z) = \bar{z}$ on the curve. The Schwarz functions associated to the boundary curves of our quadrature domain $\Omega$ have the following stronger properties. There is a function $S(z)$ that is meromorphic on an open set containing the closure of $\Omega$ that has no poles on the boundary of $\Omega$ and satisfies the identity $\bar{z} = S(z)$ on the boundary. Quadrature domains can be thought of as a generalization of the unit disc (which is a *one point* quadrature domain), and Gustafsson's conformal mapping as a generalization of the Riemann map in the $n$-connected setting.

Given a continuous function $\varphi$ on the boundary of our quadrature domain $\Omega$, we can approximate it by a rational function of $x$ and $y$ via the Stone-Weierstrass theorem since the family of such rational functions without singularities on the boundary forms an algebra of continuous functions that separates points. Writing such a rational function as a rational function of $z$ and $\bar{z}$ and replacing $\bar{z}$ by $S(z)$ produces a meromorphic function on a neighborhood of the closure of $\Omega$ that has no poles on the boundary and that agrees with the given rational function on the boundary. If we can solve the Dirichlet problem on $\Omega$ with boundary data

$$\frac{1}{(z-a)^n}$$

for fixed $a$ in $\Omega$ and positive integers $n$, then, by subtracting such solutions from the data, we would have harmonic functions that vanish on the boundary and have general pole behavior at $z = a$. We could then use these functions to subtract off the poles of our meromorphic function and obtain a solution to the Dirichlet problem with the given rational boundary data. Then we could take uniform limits and solve the Dirichlet problem for continuous boundary data $\varphi$ just like we did in the unit disc.

In case $\Omega$ is simply connected, it is possible to solve the Dirichlet problem with boundary data $(z-a)^{-n}$ using a Riemann mapping function. Let $f : \Omega \to D_1(0)$ be a Riemann map. The Green's function $G(z,w)$ for $\Omega$ is a constant times

$$\ln \left| \frac{f(z) - f(w)}{1 - \overline{f(w)}\, f(z)} \right|,$$

and derivatives

$$\frac{\partial^m}{\partial w^m} G(z,w)$$

are harmonic on $\Omega - \{w\}$, are continuous up to the boundary in $z$ and vanish on the boundary in $z$, and the singularity at $w$ is precisely of the form a constant times $(z-w)^{-m}$. One does not need to know that the Riemann map is continuous up to the boundary to see that these functions extend continuously up to the boundary in $z$ and vanish there. This follows from the fact that conformal mappings are *proper* mappings: the inverse image of a compact subset of the unit disc is a compact subset of $\Omega$.

Hence, in the simply connected case, we have a method to solve the Dirichlet problem rather analogous to the method we used in the case of the unit disc. There is something appealing about taking a close approximation to our original

domain, followed by a close approximation to the boundary data, to be able to find an elementary formula for the solution to the Dirichlet problem.

Riemann maps associated to simply connected quadrature domains can be expressed as rational combinations of $z$ and the Schwarz function, so solutions to the Dirichlet problem with rational boundary data can also be expressed as rational combinations of $z$ and the Schwarz function!

Another way to construct the Poisson kernel is to express it in terms of a normal derivative of the Green's function, which, on a simply connected quadrature domain, is also expressible in terms of a Riemann map, and hence, also expressible in terms of $z$ and the Schwarz function. It follows that the Poisson kernel of a simply connected quadrature domain is expressible in terms of $z$ and the Scwharz function. Could we do similar things in the multiply connected setting? Could we use Ahlfors maps in place of a Riemann map? Might the Poisson kernel there be expressible in terms of $z$ and a Schwarz function and the harmonic measure functions? We wonder.

## References

[1] Ahlfors, L., *Complex analysis*, 3rd Edition, McGraw Hill, New York, 1979.
[2] Bell, S., *The Cauchy transform, potential theory, and conformal mapping*, 2nd Edition, CRC Press, Boca Raton, 2016.
[3] Bell, S., *The Dirichlet and Neumann and Dirichlet-to-Neumann problems in quadrature, double quadrature, and non-quadrature domains*, Analysis and Mathematical Physics **5** (2015), 113–135.
[4] Bell, S., P. Ebenfelt, D. Khavinson, H. Shapiro, *On the classical Dirichlet problem in the plane with rational data*, J. Anal. Math. **100** (2006), 157–190.
[5] Globevnik, J., *Zero integrals on circles and characterizations of harmonic and analytic functions*, Trans. Amer. Math. Soc. **317** (1990), 313–330.
[6] Gustafsson, B., *Quadrature domains and the Schottky double*, Acta Applicandae Math. **1** (1983), 209–240.
[7] Rudin, W., *Real and Complex Analysis*, McGraw Hill, New York, 1987.
[8] Springer, G., *On Morera's theorem*, Amer. Math. Monthly **64** (1957), 323–331.
[9] Stein, E. M. and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, Princeton Press, Princeton, 2003.

Mathematics Department, Purdue University, West Lafayette, IN 47907

*Email address*: bell@math.purdue.edu

Department of Physics, Arizona State University, Tempe, AZ 85281

*Email address*:    lereyna@asu.edu